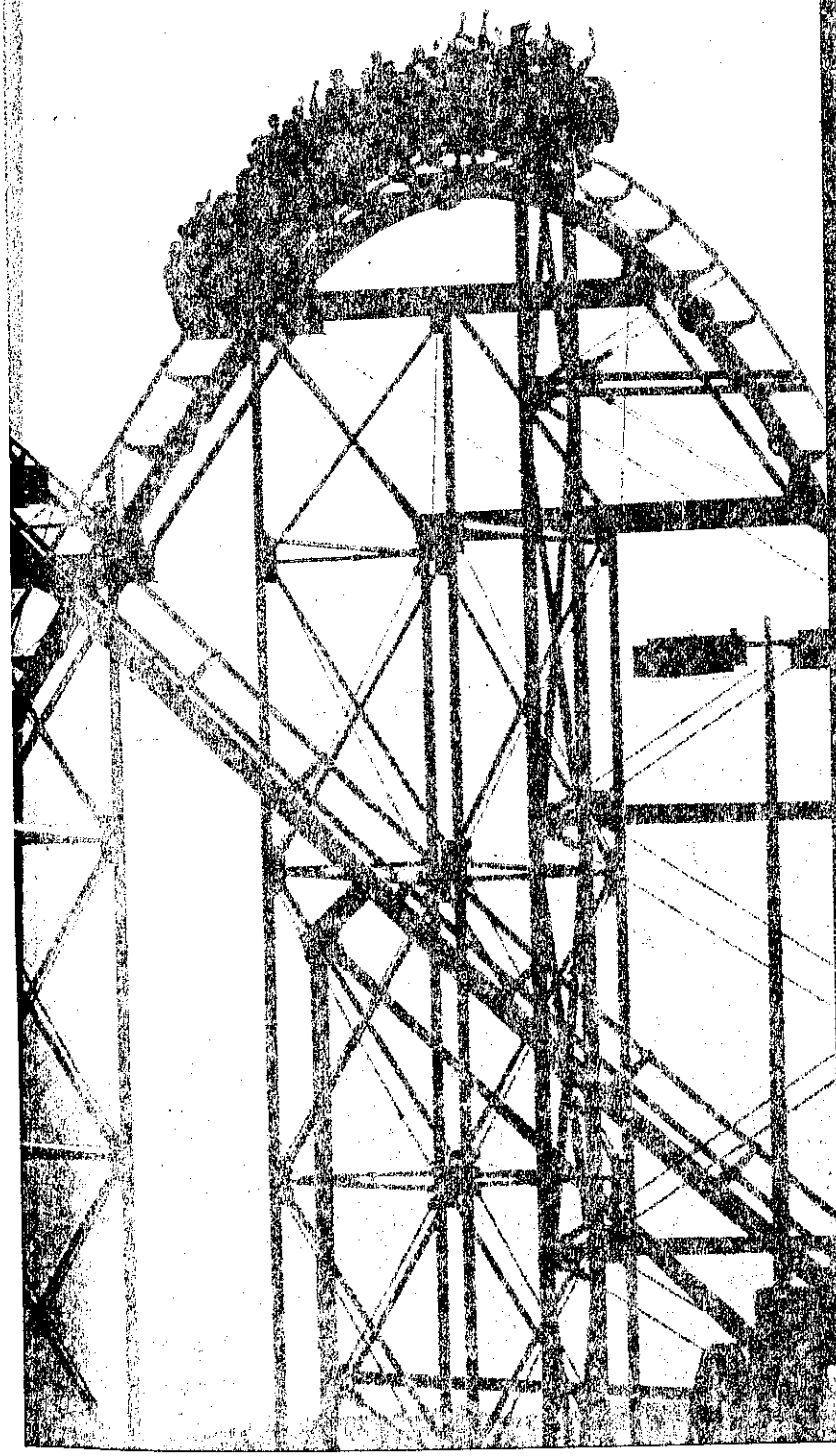
Chapter One

A LIBRARY OF FUNCTIONS



Functions are truly fundamental to mathematics. In everyday language we say, "The fuel needed to launch a rocket is a function of its payload" or "The patient's blood pressure is a function of the drugs prescribed." In each case, the word *function* expresses the idea that knowledge of one fact tells us another. In mathematics, the most important functions are those in which knowledge of one number tells us another number. If we know the length of the side of a square, its area is determined. If the circumference of a circle is known, its radius is determined.

Calculus starts with the study of functions. This chapter lays the foundation for calculus by surveying the behavior of the most common functions, including exponential, logarithmic, and trigonometric functions. We see functions represented by graphs, tables, and formulas and investigate what limits and continuity can tell us about functions.

1.1 FUNCTIONS AND CHANGE

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In December 2000, the temperatures in Chicago were unusually low over winter vacation. The daily high temperatures for December 19–28 are given in Table 1.1.

Table 1.1 *Daily high temperature in Chicago, December 19–28, 2000*

Date (December 2000)	19	20	21	22	23	24	25	26	27	28
High temperature (°F)	20	17	19	7	20	11	17	19	17	20

Although you may not have thought of something so unpredictable as temperature as being a function, the temperature *is* a function of date, because each day gives rise to one and only one high temperature. There is no formula for temperature (otherwise we would not need the weather bureau), but nevertheless the temperature does satisfy the definition of a function: Each date, t, has a unique high temperature, H, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the temperature example, the domain is the set of dates $t = \{19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$ and the range is the set of temperatures $H = \{7, 11, 17, 19, 20\}$. We call the function f and write $H = f(t)$. Notice that a function may have identical outputs for different inputs (December 20, 25, and 27, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is written $[a, b]$.

The set of numbers t such that $a < t < b$ is written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily high temperatures in Chicago can be represented by the graph in Figure 1.1, as well as by Table 1.1.

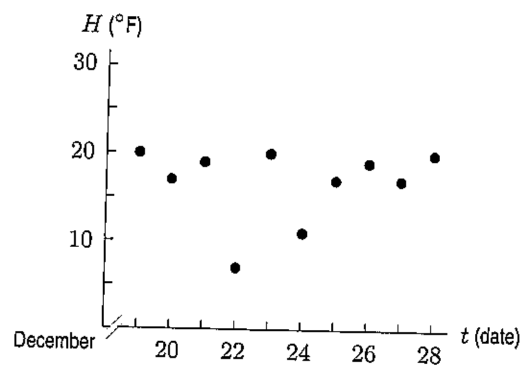


Figure 1.1: Chicago temperatures, December 2000

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed by the formula

$$C = 4T - 160$$

to a fair degree of accuracy. We write $C = f(T)$ to express the fact that we think of C as a function of T and that we have named this function f. The graph of this function is in Figure 1.2.

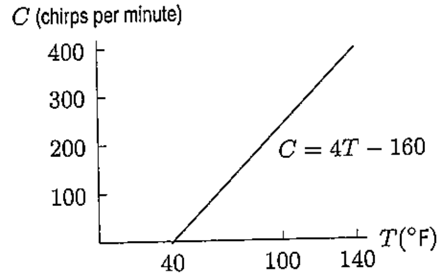


Figure 1.2: Cricket chirp rate versus temperature

Examples of Domain and Range

If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers. However, the domain of the function $g(x) = 1/x$ is all real numbers except zero, since we cannot divide by zero.

Sometimes we restrict the domain to be smaller than the largest possible set of real numbers. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to nonnegative values of x.

Example 1 The function $C = f(T)$ gives chirp rate as a function of temperature. We restrict this function to temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, 136°F. What is the domain of this function f?

Solution If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between two variables C and T, any T value is possible. However, if we think of it as a relationship between cricket chirps and temperature, then C cannot be less than 0. Since $C = 0$ leads to $0 = 4T - 160$, and so $T = 40$°F, we see that T cannot be less than 40°F. (See Figure 1.2.) In addition, we are told that the function is not defined for temperatures above 136°. Thus, for the function $C = f(T)$ we have

$$\begin{aligned} \text{Domain} &= \text{All } T \text{ values between } 40^\circ F \text{ and } 136^\circ F \\ &= \text{All } T \text{ values with } 40 \le T \le 136 \\ &= [40, 136]. \end{aligned}$$

Example 2 Find the range of the function f, given the domain from Example 1. In other words, find all possible values of the chirp rate, C, in the equation $C = f(T)$.

Solution Again, if we consider $C = 4T - 160$ simply as a mathematical relationship, its range is all real C values. However, when thinking of the meaning of $C = f(T)$ for crickets, we see that the function predicts cricket chirps per minute between 0 (at $T = 40$°F) and 384 (at $T = 136$°F). Hence,

$$\begin{aligned} \text{Range} &= \text{All } C \text{ values from 0 to 384} \\ &= \text{All } C \text{ values with } 0 \le C \le 384 \\ &= [0, 384]. \end{aligned}$$

In using the temperature to predict the chirp rate, we thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backward, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Linear Functions

The chirp-rate function, $C = f(T)$, is an example of a *linear function*. A function is linear if its slope, or rate of change, is the same at every point. The rate of change of a function that is not linear may vary from point to point.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year. So the height was a linear function of time from 1900 to 1912. If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

Since $y = f(t)$ increases with t, we say that f is an *increasing function*. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

Table 1.2 *Men's Olympic pole vault winning height (approximate)*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

This rate of increase is the *slope* of the line in Figure 1.3. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/year.}$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, 130 is the *intercept* on the vertical axis.

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130+2t$ predicts that the height in the 2000 Olympics would be 330 inches or 27 feet 6 inches, which is considerably higher than the actual value of 19 feet 4.27 inches. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 130 + 2t$ makes

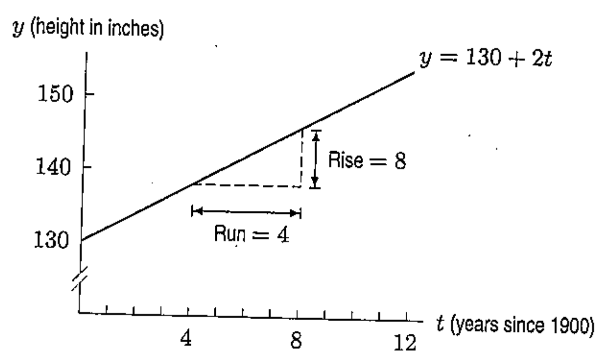


Figure 1.3: Olympic pole vault records

sense for all values of t. The graph in Figure 1.3 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

As the pole vault heights have increased over the years, the time to run the mile has decreased. If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.4t.$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.4, tells us that the world record decreased by about 0.4 seconds per year. We say that g is a *decreasing function.*

Difference Quotients and Delta Notation

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y.

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.4). Since $m = \Delta y/\Delta x$, the units of m are y-units over x-units.

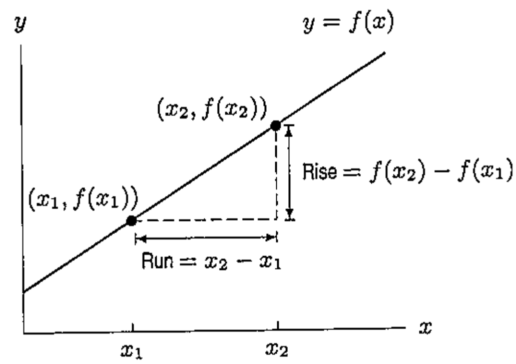


Figure 1.4: Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Families of Linear Functions

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line.

To recognize that a table of x and y values comes from a linear function, $y = b + mx$, look for differences in y-values that are constant for equally spaced x-values.

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, give a *family of functions*. All the functions in a family share certain properties—in this case, all the graphs are straight lines. The constants m and b are called *parameters*; their meaning is shown in Figures 1.5 and 1.6. Notice the greater the magnitude of m, the steeper the line.

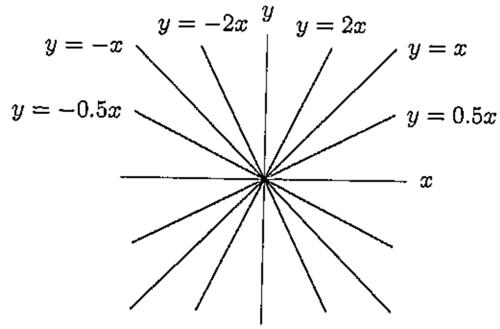


Figure 1.5: The family $y = mx$ (with $b = 0$)

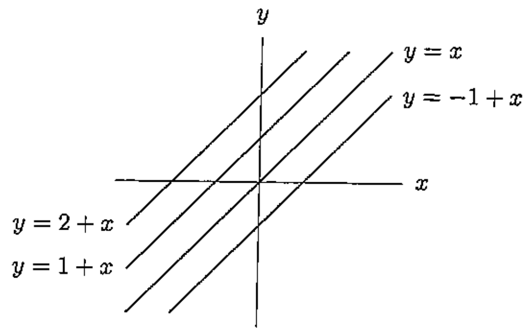


Figure 1.6: The family $y = b + x$ (with $m = 1$)

Increasing versus Decreasing Functions

The terms increasing and decreasing can be applied to other functions, not just linear ones. See Figure 1.7. In general,

> A function f is **increasing** if the values of $f(x)$ increase as x increases.
> A function f is **decreasing** if the values of $f(x)$ decrease as x increases.
>
> The graph of an *increasing* function *climbs* as we move from left to right.
> The graph of a *decreasing* function *falls* as we move from left to right.

Figure 1.7: Increasing and decreasing functions

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, the area, A, of a circle is proportional to the square of the radius, r, because

$$A = f(r) = \pi r^2.$$

> We say y is (directly) **proportional** to x if there is a nonzero constant k such that
>
> $$y = kx.$$
>
> This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = 50\left(\frac{1}{t}\right) = \frac{50}{t}.$$

Exercises and Problems for Section 1.1

Exercises

1. The population of a city, P, in millions, is a function of t, the number of years since 1950, so $P = f(t)$. Explain the meaning of the statement $f(35) = 12$ in terms of the population of this city.

2. The pollutant PCB (polychlorinated biphenyl) affects the thickness of pelican eggs. Thinking of the thickness, T, of the eggs, in mm, as a function of the concentration, P, of PCBs in ppm (parts per million), we have $T = f(P)$. Explain the meaning of $f(200)$ in terms of thickness of pelican eggs and concentration of PCBs.

3. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, a, in years.
   (a) Interpret the statement $f(5) = 6$
   (b) Sketch a possible graph of V against a. Is f an increasing or decreasing function? Explain.
   (c) Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.

For Exercises 4–6, find the equation of the line that passes through the given points.

4. $(0,0)$ and $(1,1)$
5. $(0,2)$ and $(2,3)$
6. $(-2,1)$ and $(2,3)$

For Exercises 7–9, determine the slope and the y-intercept of the line whose equation is given.

7. $7y + 12x - 2 = 0$
8. $-4y + 2x + 8 = 0$
9. $12x = 6y + 4$

10. Match the graphs in Figure 1.8 with the following equations. (Note that the x and y scales may be unequal.)

(a) $y = x - 5$	(b) $-3x + 4 = y$
(c) $5 = y$	(d) $y = -4x - 5$
(e) $y = x + 6$	(f) $y = x/2$

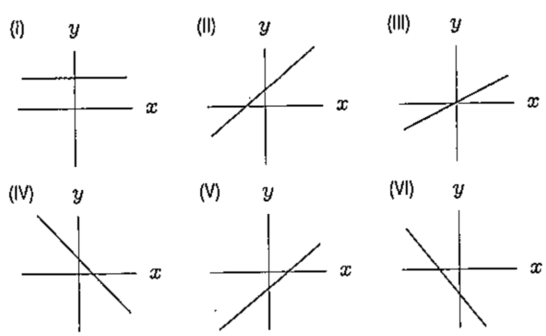


Figure 1.8

11. Match the graphs in Figure 1.9 with the following equations. (Note that the x and y scales may be unequal.)

(a) $y = -2.72x$	(b) $y = 0.01 + 0.001x$
(c) $y = 27.9 - 0.1x$	(d) $y = 0.1x - 27.9$
(e) $y = -5.7 - 200x$	(f) $y = x/3.14$

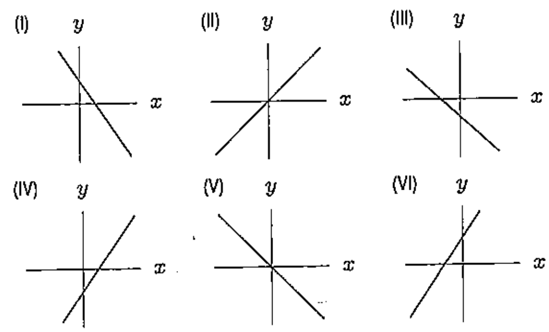


Figure 1.9

For Exercises 12–14, use the facts that parallel lines have equal slopes and that the slopes of perpendicular lines are negative reciprocals of one another.

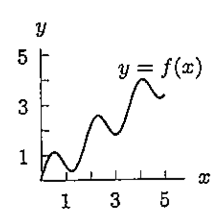
12. Find the equation of the line through the point $(2,1)$ which is perpendicular to the line $y = 5x - 3$.

13. Find the equations of the lines through the point $(1,5)$ that are parallel to and perpendicular to the line with equation $y + 4x = 7$.

14. Find the equations of the lines through the point (a,b) that are parallel and perpendicular to the line $y = mx + c$, assuming $m \neq 0$.

For Exercises 15–17, give the approximate domain and range of each function. Assume the entire graph is shown.

15.



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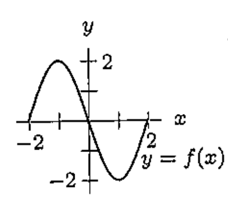

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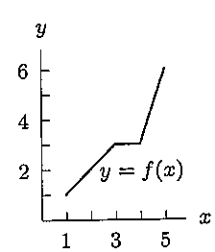


Find domain and range in Exercises 18–19.

18. $y = x^2 + 2$

19. $y = \dfrac{1}{x^2 + 2}$

20. If $f(t) = \sqrt{t^2 - 16}$, find all values of t for which $f(t)$ is a real number. Solve $f(t) = 3$.

21. If $g(x) = (4 - x^2)/(x^2 + x)$, find the domain of $g(x)$. Solve $g(x) = 0$.

In Exercises 22–24, write a formula representing the function.

22. The strength, S, of a beam is proportional to the square of its thickness, h.

23. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin.

24. The number of animal species, N, of a certain body length, l, is inversely proportional to the square of l.

Problems

25. Match each story to the most appropriate graph in Figure 1.10. Write a story for the remaining graph.

(a) During my journey, I had a flat tire. Then after fixing the flat tire, I had to speed up to avoid being late.
(b) My car broke down and I parked it at the road side.
(c) As soon as I dropped off the package, I turned around and drove home.

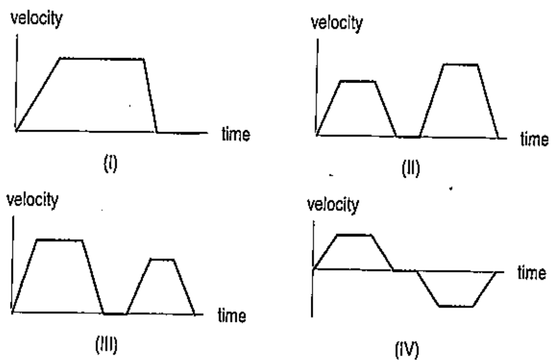


Figure 1.10

26. A car starts out slowly and then goes faster and faster until a tire blows out. Sketch a possible graph of the distance the car has traveled as a function of time.

27. A flight from Dulles Airport in Washington, DC, to LaGuardia Airport in New York City has to circle LaGuardia several times before being allowed to land. Plot a graph of the distance of the plane from Washington, DC, against time, from the moment of takeoff until landing.

28. You drive at a constant speed from Chicago to Detroit, a distance of 275 miles. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.

29. The monthly charge for a waste collection service is \$32 for 100 kg of waste and is \$48 for 180 kg of waste.

(a) Find a linear formula for the cost, C, of waste collection as a function of the number of kilograms of waste, w.
(b) What is the slope of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.
(c) What is the vertical intercept of the line found in part (a)? Give units with your answer and interpret it in terms of the cost of waste collection.

30. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount yearly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed \$90, while one using 1600 cubic feet was billed \$105.

(a) What is the charge per cubic foot?
(b) Write an equation for the total cost of a resident's water as a function of cubic feet of water used.
(c) How many cubic feet of water used would lead to a bill of \$130?

31. The graph of Fahrenheit temperature, °F, as a function of Celsius temperature, °C, is a line. You know that 212°F and 100°C both represent the temperature at which water boils. Similarly, 32°F and 0°C both represent water's freezing point.

(a) What is the slope of the graph?
(b) What is the equation of the line?
(c) Use the equation to find what Fahrenheit temperature corresponds to 20°C.
(d) What temperature is the same number of degrees in both Celsius and Fahrenheit?

32. An object is put outside on a cold day at time $t = 0$. Its temperature, $H = f(t)$, in °C, is graphed in Figure 1.11.

(a) What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.
(b) Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

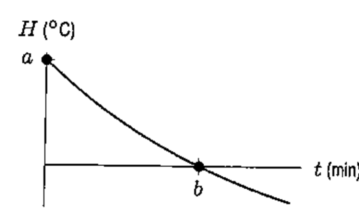


Figure 1.11

33. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. "Straight-line depreciation" assumes that the value is a linear function of time. If a \$950 refrigerator depreciates completely in seven years, find a formula for its value as a function of time.

34. The table gives the average weight, w, in pounds, of American men in their sixties for various heights, h, in inches.[1]

    (a) How do you know that the data in this table could represent a linear function?
    (b) Find weight, w, as a linear function of height, h. What is the slope of the line? What are the units for the slope?
    (c) Find height, h, as a linear function of weight, w. What is the slope of the line? What are the units for the slope?

h (inches)	68	69	70	71	72	73	74	75
w (pounds)	166	171	176	181	186	191	196	201

35. The demand function for a certain product, $q = D(p)$, is linear, where p is the price per item in dollars and q is the quantity demanded. If p increases by \$5, market research shows that q drops by two items. In addition, 100 items are purchased if the price is \$550.

    (a) Find a formula for
        (i) q as a linear function of p
        (ii) p as a linear function of q
    (b) Draw a graph with q on the horizontal axis.

36. When a cold yam is put into a hot oven to bake, the temperature of the yam rises. The rate, R (in degrees per minute), at which the temperature of the yam rises is governed by Newton's Law of Heating, which says that the rate is proportional to the temperature difference between the yam and the oven. If the oven is at 350°F and the temperature of the yam is H°F,

    (a) Write a formula giving R as a function of H.
    (b) Sketch the graph of R against H.

37. For small changes in temperature, the formula for the expansion of a metal rod under a change in temperature is:

$$l - l_0 = al_0(t - t_0),$$

where l is the length of the object at temperature t, and l_0 is the length at temperature t_0, and a is a constant which depends on the type of metal.

    (a) Express l as a linear function of t. Find the slope and vertical intercept in terms of l_0, t_0, and a.
    (b) A rod is 100 cm long at 60°F and made of a metal with $a = 10^{-5}$. Write an equation giving the length of this rod at temperature t.
    (c) What does the sign of the slope tell you about the expansion of a metal under a change in temperature?

38. A body of mass m is falling downward with velocity v. Newton's Second Law of Motion, $F = ma$, says that the net downward force, F, on the body is proportional to its downward acceleration, a. The net force, F, consists of the force due to gravity, F_g, which acts downward, minus the air resistance, F_r, which acts upward. The force due to gravity is mg, where g is a constant. Assume the air resistance is proportional to the velocity of the body.

    (a) Write an expression for the net force, F, as a function of the velocity, v.
    (b) Write a formula giving a as a function of v.
    (c) Sketch a against v.

39. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen. What do the experimental data in Table 1.3 tell you about Galileo's hypothesis? What alternative hypothesis is suggested by the two sets of data in Table 1.3 and Table 1.4?

Table 1.3

Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table 1.4

Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

1.2 EXPONENTIAL FUNCTIONS

Population Growth

The population of Mexico in the early 1980s is given in Table 1.5. To see how the population is growing, we look at the increase in population in the third column. If the population had been growing linearly, all the numbers in the third column would be the same.

[1] Adapted from "Average Weight of Americans by Height and Age", *The World Almanac* (New Jersey: Funk and Wagnalls, 1992), p. 956.

Table 1.5 *Population of Mexico (estimated), 1980–1986*

Year	Population (millions)	Change in population (millions)
1980	67.38	
1981	69.13	1.75
1982	70.93	1.80
1983	72.77	1.84
1984	74.66	1.89
1985	76.60	1.94
1986	78.59	1.99

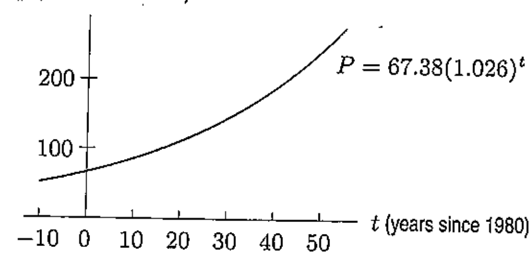


Figure 1.12: Population of Mexico (estimated): Exponential growth

Suppose we divide each year's population by the previous year's population. For example,

$$\frac{\text{Population in 1981}}{\text{Population in 1980}} = \frac{69.13 \text{ million}}{67.38 \text{ million}} = 1.026$$

$$\frac{\text{Population in 1982}}{\text{Population in 1981}} = \frac{70.93 \text{ million}}{69.13 \text{ million}} = 1.026.$$

The fact that both calculations give 1.026 shows the population grew by about 2.6% between 1980 and 1981 *and* between 1981 and 1982. Similar calculations for other years show that the population grew by a factor of about 1.026, or 2.6%, every year. Whenever we have a constant growth factor (here 1.026), we have exponential growth. The population t years after 1980 is given by the *exponential* function

$$P = 67.38(1.026)^t.$$

If we assume that the formula holds for 50 years, the population graph has the shape shown in Figure 1.12. Since the population is growing faster and faster as time goes on, the graph is bending upward; we say it is *concave up*. Even exponential functions which climb slowly at first, such as this one, eventually climb extremely quickly.

> To recognize that a table of t and P values comes from an exponential function $P = P_0a^t$, look for ratios of P values that are constant for equally spaced t values.

Concavity

We have used the term concave up[2] to describe the graph in Figure 1.12. In words:

> The graph of a function is **concave up** if it bends upward as we move left to right; it is **concave down** if it bends downward. (See Figure 1.13.) A line is neither concave up nor concave down.

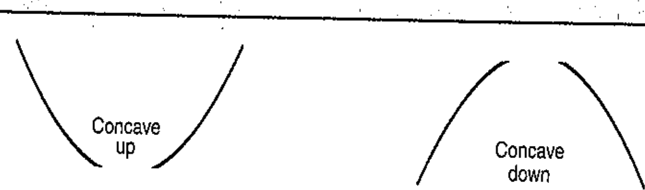


Figure 1.13: Concavity of a graph

[2]In Chapter 2 we consider concavity in more depth.

Elimination of a Drug from the Body

Now we look at a quantity which is decreasing exponentially instead of increasing. When a patient is given medication, the drug enters the bloodstream. As the drug passes through the liver and kidneys, it is metabolized and eliminated at a rate that depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since every hour the amount remaining is 60% of the previous amount, we have

$$f(0) = 250$$
$$f(1) = 250(0.6)$$
$$f(2) = (250(0.6))(0.6) = 250(0.6)^2,$$

and after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This is an *exponential decay function*. Some values of the function are in Table 1.6; its graph is in Figure 1.14.

Notice the way the function in Figure 1.14 is decreasing. Each hour a smaller quantity of the drug is removed than in the previous hour. This is because as time passes, there is less of the drug in the body to be removed. Compare this to the exponential growth in Figure 1.12, where each step upward is larger than the previous one. Notice, however, that both graphs are concave up.

Table 1.6

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

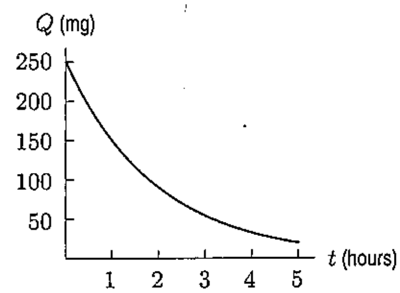


Figure 1.14: Drug elimination: Exponential decay

The General Exponential Function

> We say P is an **exponential function** of t with base a if
>
> $$P = P_0 a^t,$$
>
> where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1.
> If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

Provided $a > 0$, the largest possible domain for the exponential function is all real numbers. The reason we do not want $a \leq 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 1^t = P_0$ is then a constant function.

The value of a is closely related to the percent growth (or decay) rate. For example, if $a = 1.03$, then P is growing at 3%; if $a = 0.94$, then P is decaying at 6%.

Example 1 Suppose that $Q = f(t)$ is an exponential function of t. If $f(20) = 88.2$ and $f(23) = 91.4$:
(a) Find the base. (b) Find the growth rate. (c) Evaluate $f(25)$.

Solution (a) Let $Q = Q_0 a^t$.

Substituting $t = 20, Q = 88.2$ and $t = 23, Q = 91.4$ gives two equations for Q_0 and a:

$$88.2 = Q_0 a^{20} \quad \text{and} \quad 91.4 = Q_0 a^{23}.$$

Dividing the two equations enables us to eliminate Q_0:

$$\frac{91.4}{88.2} = \frac{Q_0 a^{23}}{Q_0 a^{20}} = a^3.$$

Solving for the base, a, gives

$$a = \left(\frac{91.4}{88.2}\right)^{1/3} = 1.012.$$

(b) Since $a = 1.012$, the growth rate is $0.012 = 1.2\%$.

(c) We want to evaluate $f(25) = Q_0 a^{25} = Q_0(1.012)^{25}$. First we find Q_0 from the equation

$$88.2 = Q_0(1.012)^{20}.$$

Solving gives $Q_0 = 69.5$. Thus,

$$f(25) = 69.5(1.012)^{25} = 93.6.$$

Half-Life and Doubling Time

Radioactive substances, such as uranium, decay exponentially. A certain percentage of the mass disintegrates in a given unit of time; the time it takes for half the mass to decay is called the *half-life* of the substance.

A well-known radioactive substance is carbon-14, which is used to date organic objects. When a piece of wood or bone was part of a living organism, it accumulated small amounts of radioactive carbon-14. Once the organism dies, it no longer picks up carbon-14. Using the half-life of carbon-14 (about 5730 years), we can estimate the age of the object. We use the following definitions:

> The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
> The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.

The Family of Exponential Functions

The formula $P = P_0 a^t$ gives a family of exponential functions with positive parameters P_0 (the initial quantity) and a (the base, or growth/decay factor). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.15 and 1.16.) All members of the family $P = P_0 a^t$ are concave up.

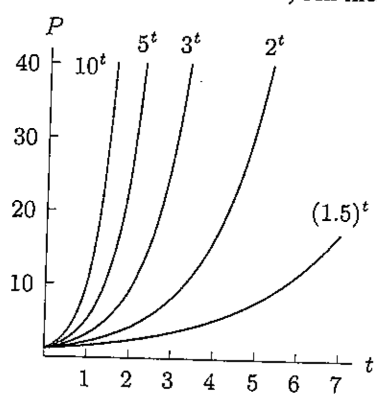


Figure 1.15: Exponential growth: $P = a^t$, for $a > 1$

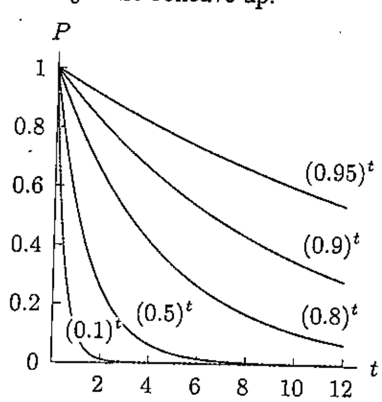


Figure 1.16: Exponential decay: $P = a^t$, for $0 < a < 1$

Example 2 Figure 1.17 is the graph of three exponential functions. What can you say about the values of the six constants, a, b, c, d, p, q?

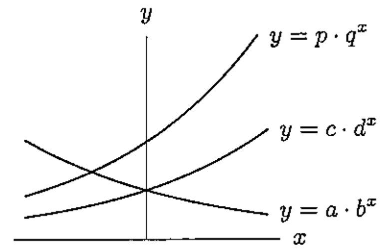


Figure 1.17

Solution All the constants are positive. Since a, c, p represent y-intercepts, we see that $a = c$ because these graphs intersect on the y-axis. In addition, $a = c < p$, since $y = p \cdot q^x$ crosses the y-axis above the other two.

Since $y = a \cdot b^x$ is decreasing, we have $0 < b < 1$. The other functions are increasing, so $1 < d$ and $1 < q$.

Exponential Functions with Base e

The most frequently used base for an exponential function is the famous number $e = 2.71828\ldots$. This base is used so often that you will find an e^x button on most scientific calculators. At first glance, this is all somewhat mysterious. Why is it convenient to use the base $2.71828\ldots$? The full answer to that question must wait until Chapter 3, where we show that many calculus formulas come out neatly when e is used as the base. We often use the following result:

> Any **exponential growth** function can be written, for some $a > 1$ and $k > 0$, in the form
>
> $$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}$$
>
> and any **exponential decay** function can be written, for some $0 < a < 1$ and $k > 0$, as
>
> $$Q = Q_0 a^t \quad \text{or} \quad Q = Q_0 e^{-kt},$$
>
> where P_0 and Q_0 are the initial quantities.
>
> We say that P and Q are growing or decaying at a *continuous*[3] *rate* of k. (For example, $k = 0.02$ corresponds to a continuous rate of 2%.)

Example 3 Convert the functions $P = e^{0.5t}$ and $Q = 5e^{-0.2t}$ into the form $y = y_0 a^t$. Use the results to explain the shape of the graphs in Figures 1.18 and 1.19.

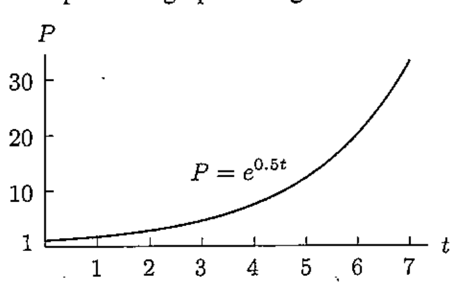


Figure 1.18: An exponential growth function

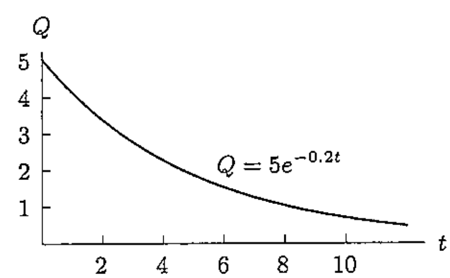


Figure 1.19: An exponential decay function

[3]The reason that k is called the continuous rate is explored in detail in Chapter 11.

Solution We have

$$P = e^{0.5t} = (e^{0.5})^t = (1.65)^t.$$

Thus, P is an exponential growth function with $P_0 = 1$ and $a = 1.65$. The function is increasing and its graph is concave up, similar to those in Figure 1.15. Also,

$$Q = 5e^{-0.2t} = 5(e^{-0.2})^t = 5(0.819)^t,$$

so Q is an exponential decay function with $Q_0 = 5$ and $a = 0.819$. The function is decreasing and its graph is concave up, similar to those in Figure 1.16.

Example 4 The quantity, Q, of a drug in a patient's body at time t is represented for positive constants S and k by the function $Q = S(1 - e^{-kt})$. For $t \geq 0$, describe how Q changes with time. What does S represent? How does the graph of Q relate to the exponential decay graph in Figure 1.19?

Solution The graph of Q is shown in Figure 1.20. Initially none of the drug is present, but the quantity increases with time. Since the graph is concave down, the quantity increases at a decreasing rate. This is realistic because as the quantity of the drug in the body increases, so does the rate at which the body excretes the drug. Thus, we expect the quantity to level off. Figure 1.20 shows that S is the saturation level.

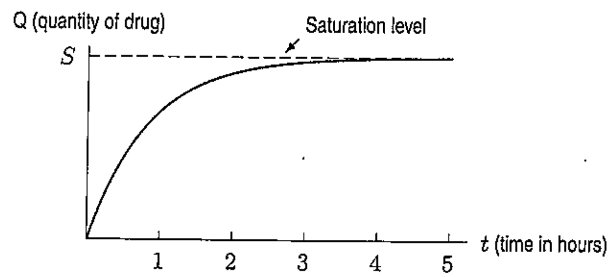


Figure 1.20: Buildup of the quantity of a drug in body

The graph in Figure 1.20 looks like an exponential decay function, but upside down. Since $Q = S - Se^{-kt}$, we have

$$S - Q = Se^{-kt}.$$

Thus, the difference between the saturation level, S, and the quantity in the blood, Q, is decaying exponentially. The graph in Figure 1.20 can be obtained from the graph in Figure 1.19 by reflection about the t-axis and moving up by a distance S.

Exercises and Problems for Section 1.2

Exercises

The functions in Exercises 1–4 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

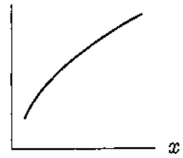
1. $P = 5(1.07)^t$
2. $P = 7.7(0.92)^t$
3. $P = 3.2e^{0.03t}$
4. $P = 15e^{-0.06t}$

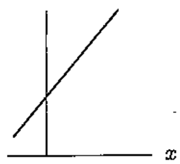
5. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.
   (a) The population increases by 50 people a year.
   (b) The population increases by 5% a year.

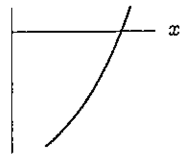
6. An air-freshener starts with 30 grams and evaporates. In each of the following cases, write a formula for the quantity, Q grams, of air-freshener remaining t days after the start and sketch a graph of the function. The decrease is:
   (a) 2 grams a day (b) 12% a day

In Exercises 7–10, decide whether the graph is concave up, concave down, or neither.

7. 

8. 

9. 

10. 

11. Identify the x-intervals on which the function graphed in Figure 1.21 is:

(a) Increasing and concave up
(b) Increasing and concave down
(c) Decreasing and concave up
(d) Decreasing and concave down

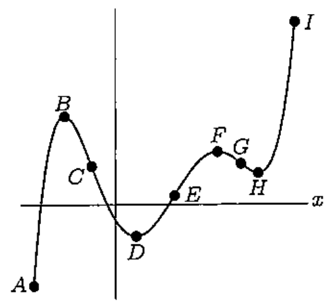


Figure 1.21

Problems

12. In 1999, the world's population reached 6 billion and was increasing at a rate of 1.3% per year. Assume that this growth rate remains constant. (In fact, the growth rate has decreased since 1987.)

(a) Write a formula for the world population (in billions) as a function of the number of years since 1999.
(b) Use your formula to estimate the population of the world in the year 2020.
(c) Sketch a graph of world population as a function of years since 1999. Use the graph to estimate the doubling time of the population of the world.

13. A photocopy machine can reduce copies to 80% of their original size. By copying an already reduced copy, further reductions can be made.

(a) If a page is reduced to 80%, what percent enlargement is needed to return it to its original size?
(b) Estimate the number of times in succession that a page must be copied to make the final copy less than 15% of the size of the original.

14. (a) Niki invested \$10,000 in the stock market. The investment was a loser, declining in value 10% per year each year for 10 years. How much was the investment worth after 10 years?
(b) After 10 years, the stock began to gain value at 10% per year. After how long will the investment regain its initial value (\$10,000)?

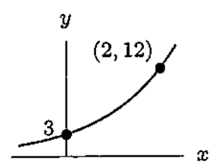
15. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

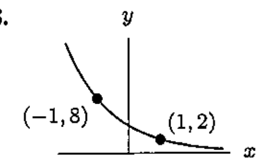
(a) Graph the total number of people who have tried such a product against time.
(b) What do you know about the concavity of the graph?

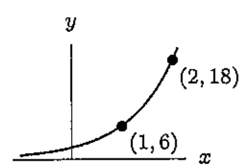
16. Sketch reasonable graphs for the following. Pay particular attention to the concavity of the graphs.

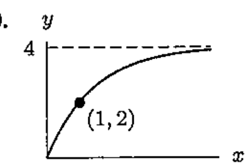
(a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.
(b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.

Give a possible formula for the functions in Problems 17–20.

17. 

18. 

19. 

20. 

In Problems 21–22, let $f(t) = Q_0 a^t = Q_0(1+r)^t$.

(a) Find the base, a.
(b) Find the percentage growth rate, r.

21. $f(5) = 75.94$ and $f(7) = 170.86$

22. $f(0.02) = 25.02$ and $f(0.05) = 25.06$

Write the functions in Problems 23–26 in the form $P = P_0 a^t$. Which represent exponential growth and which represent exponential decay?

23. $P = 15e^{0.25t}$

24. $P = 2e^{-0.5t}$

25. $P = P_0 e^{0.2t}$

26. $P = 7e^{-\pi t}$

27. (a) A population, P, grows at a continuous rate of 2% a year and starts at 1 million. Write P in the form $P = P_0e^{kt}$, with P_0, k constants.
(b) Plot the population in part (a) against time.

28. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially by 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

29. During 1988, Nicaragua's inflation rate averaged 1.3% a day. This means that, on average, prices went up by 1.3% from one day to the next.

(a) By what percentage did Nicaraguan prices increase in June of 1988?
(b) What was Nicaragua's annual inflation rate during 1988?

30. (a) The half-life of radium-226 is 1620 years. Write a formula for the quantity, Q, of radium left after t years, if the initial quantity is Q_0.
(b) What percentage of the original amount of radium is left after 500 years?

31. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 1990?

32. A certain region has a population of 10,000,000 and an annual growth rate of 2%. Estimate the doubling time by guessing and checking.

33. Estimate graphically the doubling time of the exponentially growing population shown in Figure 1.22. Check that the doubling time is independent of where you start on the graph. Show algebraically that if $P = P_0a^t$ doubles between time t and time $t + d$, then d is the same number for any t.

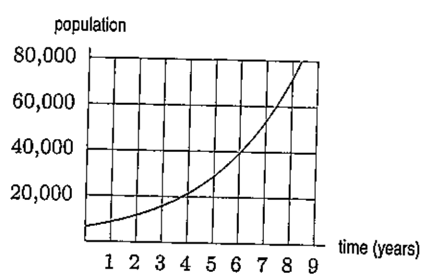


Figure 1.22

34. Aircrafts require longer takeoff distances, called takeoff rolls, at high altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

35. Each of the functions g, h, k in Table 1.7 is increasing, but each increases in a different way. Which of the graphs in Figure 1.23 best fits each function?

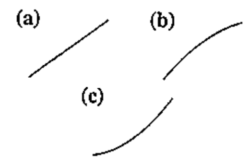


Figure 1.23

Table 1.7

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

36. (a) Which (if any) of the functions in the following table could be linear? Find formulas for those functions.
(b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
−2	12	16	37
−1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

37. Match the functions $h(s)$, $f(s)$, and $g(s)$, whose values are in Table 1.8, with the formulas

$$y = a(1.1)^s\,,\quad y = b(1.05)^s\,,\quad y = c(1.03)^s,$$

assuming a, b, and c are constants. Note that the function values have been rounded to two decimal places.

Table 1.8

s	$h(s)$	s	$f(s)$	s	$g(s)$
2	1.06	1	2.20	3	3.47
3	1.09	2	2.42	4	3.65
4	1.13	3	2.66	5	3.83
5	1.16	4	2.93	6	4.02
6	1.19	5	3.22	7	4.22

38. Table 1.9 shows some values of a linear function f and an exponential function g. Find exact values (not decimal approximations) for each of the missing entries.

Table 1.9

x	0	1	2	3	4
$f(x)$	10	?	20	?	?
$g(x)$	10	?	20	?	?

39. The median price, P, of a home rose from \$50,000 in 1970 to \$100,000 in 1990. Let t be the number of years since 1970.

(a) Assume the increase in housing prices has been linear. Give an equation for the line representing price, P, in terms of t. Use this equation to complete column (a) of Table 1.10. Use units of \$1000.

(b) If instead the housing prices have been rising exponentially, find an equation of the form $P = P_0a^t$ to represent housing prices. Complete column (b) of Table 1.10.

(c) On the same set of axes, sketch the functions represented in column (a) and column (b) of Table 1.10.

(d) Which model for the price growth do you think is more realistic?

Table 1.10

t	(a) Linear growth price in \$1000 units	(b) Exponential growth price in \$1000 units
0	50	50
10		
20	100	100
30		
40		

1.3 NEW FUNCTIONS FROM OLD

Shifts and Stretches

The graph of a constant multiple of a given function is easy to visualize: each y-value is stretched or shrunk by that multiple. For example, consider the function $f(x)$ and its multiples $y = 3f(x)$ and $y = -2f(x)$. Their graphs are shown in Figure 1.24. The factor 3 in the function $y = 3f(x)$ stretches each $f(x)$ value by multiplying it by 3; the factor -2 in the function $y = -2f(x)$ stretches $f(x)$ by multiplying by 2 and reflects it about the x-axis. You can think of the multiples of a given function as a family of functions.

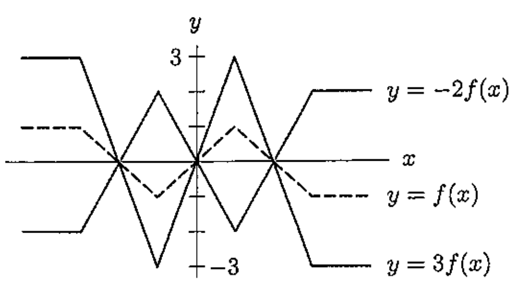


Figure 1.24: Multiples of the function $f(x)$

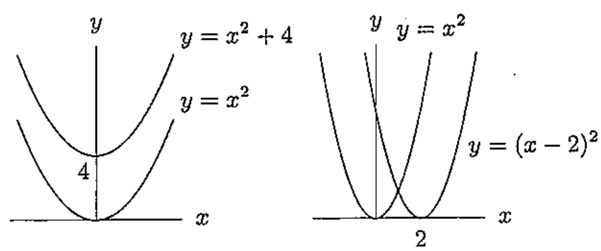


Figure 1.25: Graphs of $y = x^2$ with $y = x^2 + 4$ and $y = (x-2)^2$

It is also easy to create families of functions by shifting graphs. For example, $y - 4 = x^2$ is the same as $y = x^2 + 4$, which is the graph of $y = x^2$ shifted up by 4. Similarly, $y = (x-2)^2$ is the graph of $y = x^2$ shifted right by 2. (See Figure 1.25.)

- Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph about the x-axis, in addition to shrinking or stretching.
- Replacing y by $(y - k)$ moves a graph up by k (down if k is negative).
- Replacing x by $(x - h)$ moves a graph to the right by h (to the left if h is negative).

Composite Functions

If oil is spilled from a tanker, the area of the oil slick grows with time. Suppose that the oil slick is always a perfect circle. Then the area, A, of the oil slick is a function of its radius, r:

$$A = f(r) = \pi r^2.$$

The radius is also a function of time, because the radius increases as more oil spills. Thus, the area, being a function of the radius, is also a function of time. If, for example, the radius is given by

$$r = g(t) = 1 + t,$$

then the area is given as a function of time by substitution:

$$A = \pi r^2 = \pi(1+t)^2.$$

We are thinking of A as a *composite function* or a "function of a function," which is written

$$A = \underbrace{f(g(t))}_{\substack{\text{Composite function;}\\ f \text{ is outside function,}\\ g \text{ is inside function}}} = \pi(g(t))^2 = \pi(1+t)^2.$$

To calculate A using the formula $\pi(1+t)^2$, the first step is to find $1+t$, and the second step is to square and multiply by π. The first step corresponds to the inside function $g(t) = 1+t$, and the second step corresponds to the outside function $f(r) = \pi r^2$.

Example 1 If $f(x) = x^2$ and $g(x) = x+1$, find each of the following:
(a) $f(g(2))$ (b) $g(f(2))$ (c) $f(g(x))$ (d) $g(f(x))$

Solution
(a) Since $g(2) = 3$, we have $f(g(2)) = f(3) = 9$.
(b) Since $f(2) = 4$, we have $g(f(2)) = g(4) = 5$. Notice that $f(g(2)) \neq g(f(2))$.
(c) $f(g(x)) = f(x+1) = (x+1)^2$.
(d) $g(f(x)) = g(x^2) = x^2 + 1$. Again, notice that $f(g(x)) \neq g(f(x))$.

Example 2 Express each of the following functions as a composition:

(a) $h(t) = (1+t^3)^{27}$ (b) $k(y) = e^{-y^2}$ (c) $l(y) = -(e^y)^2$

Solution
In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

(a) For $(1+t^3)^{27}$, the first stage is cubing and adding 1, so an inside function is $g(t) = 1+t^3$. The second stage is taking the 27^{th} power, so an outside function is $f(y) = y^{27}$. Then

$$f(g(t)) = f(1+t^3) = (1+t^3)^{27}.$$

In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1+y)^{27}$ is another possibility.

(b) To calculate e^{-y^2} we square y, take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

$$f(g(y)) = e^{-y^2}.$$

(c) To calculate $-(e^y)^2$, we find e^y, square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

$$g(f(y)) = -(e^y)^2.$$

Since parts (b) and (c) give different answers, we see the order in which functions are composed is important.

Odd and Even Functions: Symmetry

There is a certain symmetry apparent in the graphs of $f(x) = x^2$ and $g(x) = x^3$ in Figure 1.26. For each point (x, x^2) on the graph of f, the point $(-x, x^2)$ is also on the graph; for each point (x, x^3) on the graph of g, the point $(-x, -x^3)$ is also on the graph. The graph of $f(x) = x^2$ is symmetric about the y-axis, whereas the graph of $g(x) = x^3$ is symmetric about the origin. The graph of any polynomial involving only even powers of x has symmetry about the y-axis, while polynomials with only odd powers of x are symmetric about the origin. Consequently, any functions with these symmetry properties are called *even* and *odd*, respectively.

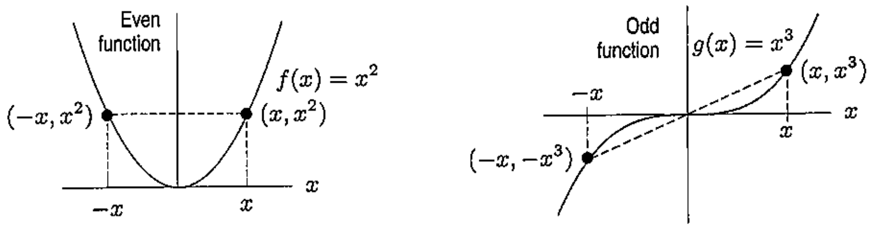


Figure 1.26: Symmetry of even and odd functions

> For any function f,
> f is an **even** function if $f(-x) = f(x)$ for all x.
> f is an **odd** function if $f(-x) = -f(x)$ for all x.

For example, $g(x) = e^{x^2}$ is even and $h(x) = x^{1/3}$ is odd. However, many functions do not have any symmetry and are neither even nor odd.

Inverse Functions

On August 18, 1989, the runner Arturo Barrios of Mexico set a world record for the 10,000-meter race. His times, in seconds, at 2000-meter intervals are recorded in Table 1.11, where $t = f(d)$ is the number of seconds Barrios took to complete the first d meters of the race. For example, Barrios ran the first 4000 meters in 650.10 seconds, so $f(4000) = 650.10$. The function f was useful to athletes planning to compete with Barrios.

Let us now change our point of view and ask for distances rather than times. If we ask how far Barrios ran during the first 650.10 seconds of his race, the answer is clearly 4000 meters. Going backward in this way from numbers of seconds to numbers of meters gives f^{-1}, the *inverse function*[4] of f. We write $f^{-1}(650.10) = 4000$. Thus, $f^{-1}(t)$ is the number of meters that Barrios ran during the first t seconds of his race. See Table 1.12 which contains values of f^{-1}.

The independent variable for f is the dependent variable for f^{-1}, and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \leq d \leq 10000$, which is the range of f^{-1}. The range of f is all times t, such that $0 \leq t \leq 1628.23$, which is the domain of f^{-1}.

Table 1.11 *Barrios's running time*

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	325.90
4000	650.10
6000	975.50
8000	1307.00
10000	1628.23

Table 1.12 *Distance run by Barrios*

t (seconds)	$d = f^{-1}(t)$ (meters)
0.00	0
325.90	2000
650.10	4000
975.50	6000
1307.00	8000
1628.23	10000

[4]The notation f^{-1} represents the inverse function, which is not the same as the reciprocal, $1/f$.

Which Functions Have Inverses?

If a function has an inverse, we say it is *invertible*. Let's look at a function which is not invertible. Consider the flight of the Mercury spacecraft *Freedom 7*, which carried Alan Shepard, Jr. into space in May 1961. Shepard was the first American to journey into space. After launch, his spacecraft rose to an altitude of 116 miles, and then came down into the sea. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why not, try to decide on a value for $f^{-1}(100)$, which should be the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.27.)

The reason the altitude function does not have an inverse is that the altitude has the same value for two different times. The reason the Barrios time function did have an inverse is that each running time, t, corresponds to a unique distance, d.

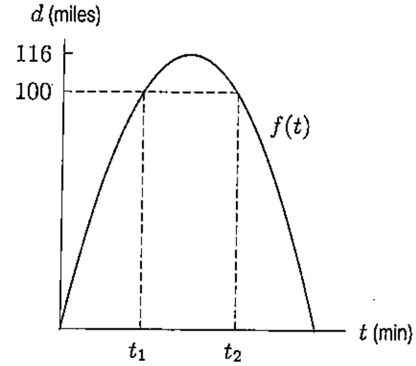


Figure 1.27: Two times, t_1 and t_2, at which altitude of spacecraft is 100 miles

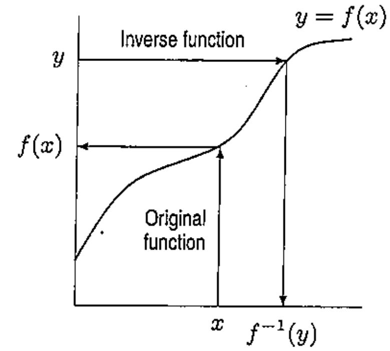


Figure 1.28: A function which has an inverse

Figure 1.28 suggests when an inverse exists. The original function, f, takes us from an x-value to a y-value, as shown in Figure 1.28. Since having an inverse means there is a function going from a y-value to an x-value, the crucial question is whether we can get back. In other words, does each y-value correspond to a unique x-value? If so, there's an inverse; if not, there is not. This principle may be stated geometrically, as follows:

> A function has an inverse if (and only if) its graph intersects any horizontal line at most once.

For example, the function $f(x) = x^2$ does not have an inverse because many horizontal lines intersect the parabola twice.

Definition of an Inverse Function

If the function f is invertible, its inverse is defined as follows:

$$f^{-1}(y) = x \quad \text{means} \quad y = f(x).$$

Formulas for Inverse Functions

If a function is defined by a formula, it is sometimes possible to find a formula for the inverse function. In Section 1.1, we looked at the snow tree cricket, whose chirp rate, C, in chirps per minute, is approximated at the temperature, T, in degrees Fahrenheit, by the formula

$$C = f(T) = 4T - 160.$$

So far we have used this formula to predict the chirp rate from the temperature. But it is also possible to use this formula backward to calculate the temperature from the chirp rate.

Example 3 Find the formula for the function giving temperature in terms of the number of cricket chirps per minute; that is, find the inverse function f^{-1} such that

$$T = f^{-1}(C).$$

Solution Since C is an increasing function, f is invertible. We know $C = 4T - 160$. We solve for T, giving

$$T = \frac{C}{4} + 40,$$

so

$$f^{-1}(C) = \frac{C}{4} + 40.$$

Graphs of Inverse Functions

The function $f(x) = x^3$ is increasing everywhere and so has an inverse. To find the inverse, we solve

$$y = x^3$$

for x, giving

$$x = y^{1/3}.$$

The inverse function is

$$f^{-1}(y) = y^{1/3}$$

or, if we want to call the independent variable x,

$$f^{-1}(x) = x^{1/3}.$$

The graphs of $y = x^3$ and $y = x^{1/3}$ are shown in Figure 1.29. Notice that these graphs are the reflections of one another about the line $y = x$. For example, $(8, 2)$ is on the graph of $y = x^{1/3}$ because $2 = 8^{1/3}$, and $(2, 8)$ is on the graph of $y = x^3$ because $8 = 2^3$. The points $(8, 2)$ and $(2, 8)$ are reflections of one another about the line $y = x$. In general, if the x- and y-axes have the same scales:

> The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

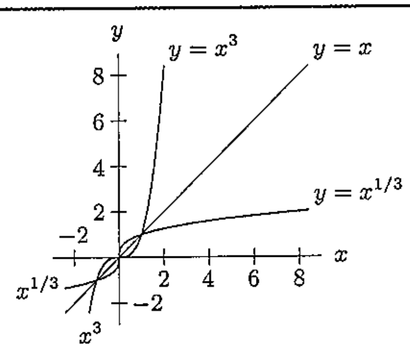


Figure 1.29: Graphs of inverse functions, $y = x^3$ and $y = x^{1/3}$, are reflections about the line $y = x$

Exercises and Problems for Section 1.3

Exercises

For the functions f and g in Exercises 1–4, find

(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(g(x))$
(d) $g(f(x))$ (e) $f(t)g(t)$

1. $f(x) = x^2, g(x) = x + 1$
2. $f(x) = \sqrt{x+4}, g(x) = x^2$
3. $f(x) = e^x, g(x) = x^2$
4. $f(x) = 1/x, g(x) = 3x + 4$

In Exercises 5–8, use Figure 1.30 to graph the functions.

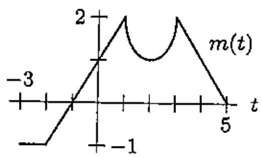


Figure 1.30

5. $n(t) = m(t) + 2$

6. $p(t) = m(t - 1)$

7. $k(t) = m(t + 1.5)$

8. $w(t) = m(t - 0.5) - 2.5$

In Exercises 9–13, use Figure 1.31 to graph the function.

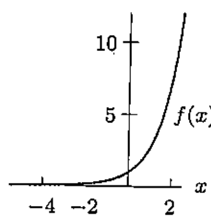


Figure 1.31

9. $f(-x)$

10. $f(x) + 5$

11. $f(x + 5)$

12. $5f(x)$

13. $f(5x)$

14. For $g(x) = x^2 + 2x + 3$, find and simplify:
(a) $g(2 + h)$ (b) $g(2)$
(c) $g(2 + h) - g(2)$

15. If $f(x) = x^2 + 1$, find and simplify:
(a) $f(t + 1)$ (b) $f(t^2 + 1)$ (c) $f(2)$
(d) $2f(t)$ (e) $[f(t)]^2 + 1$

16. For $f(n) = 3n^2 - 2$ and $g(n) = n + 1$, find and simplify:
(a) $f(n) + g(n)$
(b) $f(n)g(n)$
(c) The domain of $f(n)/g(n)$
(d) $f(g(n))$
(e) $g(f(n))$

Simplify the quantities in Exercises 17–20 using $m(z) = z^2$.

17. $m(z + 1) - m(z)$

18. $m(z + h) - m(z)$

19. $m(z) - m(z - h)$

20. $m(z + h) - m(z - h)$

21. Let p be the price of an item and q be the number of items sold at that price, where $q = f(p)$. What do the following quantities mean in terms of prices and quantities sold?
(a) $f(25)$ (b) $f^{-1}(30)$

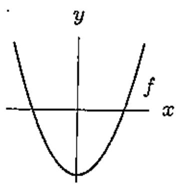
22. Let $C = f(A)$ be the cost, in dollars, of building a store of area A square feet. In terms of cost and square feet, what do the following quantities represent?
(a) $f(10{,}000)$ (b) $f^{-1}(20{,}000)$

23. Let $f(x)$ be the temperature (°F) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(75)$ in practical terms?

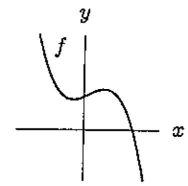
24. Let $m = f(A)$ be the minimum annual gross income, in thousands of dollars, needed to obtain a 30-year home mortgage loan of A thousand dollars at an interest rate of 9%. What do the following quantities represent in terms of the income needed for a loan?
(a) $f(100)$ (b) $f^{-1}(75)$

For Exercises 25–26, decide if the function $y = f(x)$ is invertible.

25.



26.



27. (a) Use Figure 1.32 to estimate $f^{-1}(2)$.
(b) Sketch a graph of f^{-1} on the same axes.

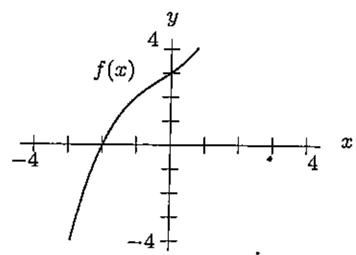
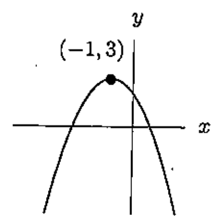


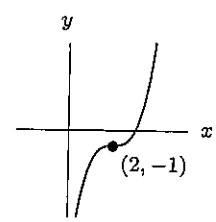
Figure 1.32

Find possible formulas for the graphs in Exercises 28–29 using shifts of x^2 or x^3.

28.



29.



Are the functions in Exercises 30–37 even, odd, or neither?

30. $f(x) = x^6 + x^3 + 1$

31. $f(x) = x^3 + x^2 + x$

32. $f(x) = x^4 - x^2 + 3$

33. $f(x) = x^3 + 1$

34. $f(x) = 2x$

35. $f(x) = e^{x^2 - 1}$

36. $f(x) = x(x^2 - 1)$

37. $f(x) = e^x - x$

Problems

38. (a) Write an equation for a graph obtained by vertically stretching the graph of $y = x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit. Sketch it.
(b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
(c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

For Problems 39–42, decide if the function f is invertible.

39. $f(t)$ is the number of customers in Macy's department store at t minutes past noon on December 18, 2000.

40. $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.

41. $f(x)$ is the volume in liters of x kg of water at 4°C.

42. $f(w)$ is the cost of mailing a letter weighing w grams.

For Problems 43–48, use the graphs in Figure 1.33.

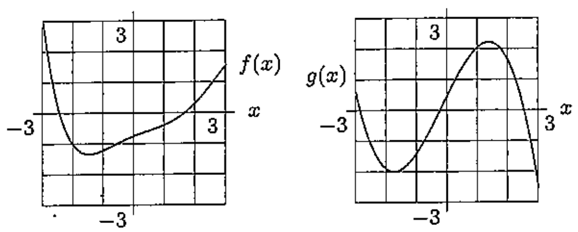


Figure 1.33

43. Estimate $f(g(1))$.

44. Estimate $g(f(2))$.

45. Estimate $f(f(1))$.

46. Graph $f(g(x))$.

47. Graph $g(f(x))$.

48. Graph $f(f(x))$.

For Problems 49–50, determine functions f and g such that $h(x) = f(g(x))$. [Note: There is more than one correct answer. Do not choose $f(x) = x$ or $g(x) = x$.]

49. $h(x) = (x+1)^3$

50. $h(x) = x^3 + 1$

51. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves where $n = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.

52. The cost of producing q articles is given by the function $C = f(q) = 100 + 2q$.

(a) Find a formula for the inverse function.
(b) Explain in practical terms what the inverse function tells you.

53. A kilogram weighs about 2.2 pounds.

(a) Write a formula for the function, f, which gives an object's mass in kilograms, k, as a function of its weight in pounds, p.
(b) Find a formula for the inverse function of f. What does this inverse function tell you, in practical terms?

54. Complete the following table with values for the functions f, g, and h, given that:

(a) f is an even function.
(b) g is an odd function.
(c) h is the composition $h(x) = g(f(x))$.

x	$f(x)$	$g(x)$	$h(x)$
−3	0	0	
−2	2	2	
−1	2	2	
0	0	0	
1			
2			
3			

1.4 LOGARITHMIC FUNCTIONS

In Section 1.2, we approximated the population of Mexico (in millions) by the function

$$P = f(t) = 67.38(1.026)^t,$$

where t is the number of years since 1980. Now suppose that instead of calculating the population at time t, we ask when the population will reach 200 million. We want to find the value of t for which

$$200 = f(t) = 67.38(1.026)^t.$$

We use logarithms to solve for a variable in an exponent.

Logarithms to Base 10 and to Base e

We define the *logarithm* function, $\log_{10} x$, to be the inverse of the exponential function, 10^x, as follows:

> The **logarithm** to base 10 of x, written $\log_{10} x$, is the power of 10 we need to get x. In other words,
>
> $$\log_{10} x = c \quad \text{means} \quad 10^c = x.$$
>
> We often write $\log x$ in place of $\log_{10} x$.

The other frequently used base is e. The logarithm to base e is called the *natural logarithm* of x, written $\ln x$ and defined to be the inverse function of e^x, as follows:

> The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,
>
> $$\ln x = c \quad \text{means} \quad e^c = x.$$

Values of $\log x$ are in Table 1.13. Because no power of 10 gives 0, $\log 0$ is undefined. The graph of $y = \log x$ is shown in Figure 1.34. The domain of $y = \log x$ is positive real numbers; the range is all real numbers. In contrast, the inverse function $y = 10^x$ has domain all real numbers and range all positive real numbers. The graph of $y = \log x$ has a vertical asymptote at $x = 0$, whereas $y = 10^x$ has a horizontal asymptote at $y = 0$.

One big difference between $y = 10^x$ and $y = \log x$ is that the exponential function grows extremely quickly whereas the log function grows extremely slowly. However, $\log x$ does go to infinity, albeit slowly, as x increases. Since $y = \log x$ and $y = 10^x$ are inverse functions, the graphs of the two functions are reflections of one another about the line $y = x$, provided the scales along the x- and y-axes are equal.

Table 1.13 *Values for* $\log x$ *and* 10^x

x	$\log x$
0	undefined
1	0
2	0.3
3	0.5
4	0.6
⋮	⋮
10	1

x	10^x
0	1
1	10
2	100
3	10^3
4	10^4
⋮	⋮
10	10^{10}

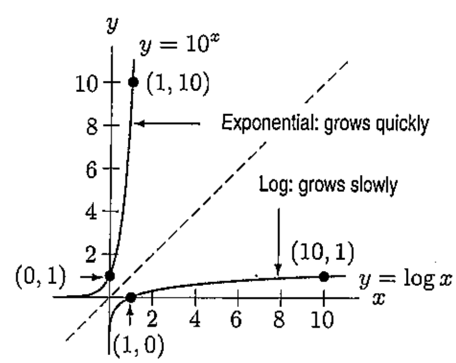


Figure 1.34: Graphs of $\log x$ and 10^x

The graph of $y = \ln x$ in Figure 1.35 has roughly the same shape as the graph of $y = \log x$. The x-intercept is $x = 1$, since $\ln 1 = 0$.The graph of $y = \ln x$ also climbs very slowly as x increases.

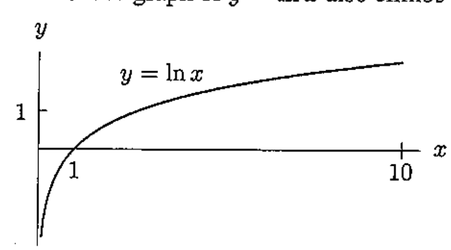


Figure 1.35: Graph of the natural logarithm

The following properties of logarithms may be deduced from the properties of exponents:

Properties of Logarithms

Note that $\log x$ and $\ln x$ are not defined when x is negative or 0.

1. $\log(AB) = \log A + \log B$
2. $\log\left(\frac{A}{B}\right) = \log A - \log B$
3. $\log(A^p) = p\log A$
4. $\log(10^x) = x$
5. $10^{\log x} = x$

1. $\ln(AB) = \ln A + \ln B$
2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\ln(A^p) = p\ln A$
4. $\ln e^x = x$
5. $e^{\ln x} = x$

In addition, $\log 1 = 0$ because $10^0 = 1$, and $\ln 1 = 0$ because $e^0 = 1$.

Solving Equations Using Logarithms

Logs are frequently useful when we have to solve for unknown exponents, as in the next examples.

Example 1 Find t such that $2^t = 7$.

Solution First, notice that we expect t to be between 2 and 3 (because $2^2 = 4$ and $2^3 = 8$). To calculate t, we take logs to base 10. (Natural logs could also be used.)

$$\log(2^t) = \log 7.$$

Then use the third property of logs, which says $\log(2^t) = t\log 2$, and get:

$$t\log 2 = \log 7.$$

Using a calculator to find the logs gives

$$t = \frac{\log 7}{\log 2} \approx 2.81.$$

Example 2 Find when the population of Mexico reaches 200 million by solving $200 = 67.38(1.026)^t$.

Solution Dividing both sides of the equation by 67.38, we get

$$\frac{200}{67.38} = (1.026)^t.$$

Now take logs of both sides:

$$\log\left(\frac{200}{67.38}\right) = \log(1.026^t).$$

Using the fact that $\log(A^t) = t\log A$, we get

$$\log\left(\frac{200}{67.38}\right) = t\log(1.026).$$

Solving this equation using a calculator to find the logs, we get

$$t = \frac{\log(200/67.38)}{\log(1.026)} \approx 42.4 \text{ years}$$

which is between $t = 42$ and $t = 43$. This value of t corresponds to the year 2022.

Example 3 The release of chlorofluorocarbons used in air conditioners and in household sprays (hair spray, shaving cream, etc.) destroys the ozone in the upper atmosphere. Currently, the amount of ozone, Q, is decaying exponentially at a continuous rate of 0.25% per year. What is the half-life of ozone?

Solution We want to find how long it takes for half the ozone to disappear. If Q_0 is the initial quantity of ozone, then

$$Q = Q_0 e^{-0.0025t}.$$

We want to find T, the value of t making $Q = Q_0/2$, that is,

$$Q_0 e^{-0.0025T} = \frac{Q_0}{2}.$$

Dividing by Q_0 and then taking natural logs yields

$$\ln\left(e^{-0.0025T}\right) = -0.0025T = \ln\left(\frac{1}{2}\right) \approx -0.6931,$$

so

$$T \approx 277 \text{ years}.$$

The half-life of ozone is about 277 years.

In Example 3 the decay rate was given. However, in many situations where we expect to find exponential growth or decay, the rate is not given. To find it, we must know the quantity at two different times and then solve for the growth or decay rate, as in the next example.

Example 4 The population of Kenya was 19.5 million in 1984 and 21.2 million in 1986. Assuming it increases exponentially, find a formula for the population of Kenya as a function of time.

Solution If we measure the population, P, in millions and time, t, in years since 1984, we can say

$$P = P_0 e^{kt} = 19.5e^{kt},$$

where $P_0 = 19.5$ is the initial value of P. We find k by using the fact that $P = 21.2$ when $t = 2$, so

$$21.2 = 19.5e^{k\cdot 2}.$$

To find k, we divide both sides by 19.5, giving

$$\frac{21.2}{19.5} = 1.087 = e^{2k}.$$

Now take natural logs of both sides:

$$\ln(1.087) = \ln(e^{2k}).$$

Using a calculator and the fact that $\ln(e^{2k}) = 2k$, this becomes

$$0.0834 = 2k.$$

So

$$k \approx 0.042,$$

and therefore

$$P = 19.5e^{0.042t}.$$

Since $k = 0.042 = 4.2\%$, the population of Kenya was growing at a continuous rate of 4.2% per year.

In Example 4 we chose to use e for the base of the exponential function representing Kenya's population, making clear that the continuous growth rate was 4.2%. If we had wanted to emphasize the annual growth rate, we could have expressed the exponential function in the form $P = P_0 a^t$.

Example 5 Give a formula for the inverse of the following function (that is, solve for t in terms of P):

$$P = f(t) = 67.38(1.026)^t.$$

Solution We want a formula expressing t as a function of P. Take logs:

$$\log P = \log(67.38(1.026)^t).$$

Since $\log(AB) = \log A + \log B$, we have

$$\log P = \log 67.38 + \log((1.026)^t).$$

Now use $\log(A^t) = t \log A$:

$$\log P = \log 67.38 + t \log 1.026.$$

Solve for t in two steps, using a calculator at the final stage:

$$t \log 1.026 = \log P - \log 67.38$$

$$t = \frac{\log P}{\log 1.026} - \frac{\log 67.38}{\log 1.026} \approx 89.7 \log P - 164.0.$$

Thus,

$$f^{-1}(P) = 89.7 \log P - 164.0.$$

Note that

$$f^{-1}(200) = 89.7(\log 200) - 164.0 \approx (89.7)(2.301) - 164.0 \approx 42.4,$$

which agrees with the result of Example 2.

Exercises and Problems for Section 1.4

Exercises

Simplify the expressions in Exercises 1–6 completely.

1. $e^{\ln(1/2)}$
2. $10^{\log(AB)}$
3. $5e^{\ln(A^2)}$
4. $\ln(e^{2AB})$
5. $\ln(1/e) + \ln(AB)$
6. $2\ln\left(e^A\right) + 3\ln B^e$

For Exercises 7–17, solve for x using logs.

7. $3^x = 11$
8. $17^x = 2$
9. $20 = 50(1.04)^x$
10. $4 \cdot 3^x = 7 \cdot 5^x$
11. $2^x = e^{x+1}$
12. $50 = 600e^{-0.4x}$
13. $2e^{3x} = 4e^{5x}$
14. $7^{x+2} = e^{17x}$
15. $10^{x+3} = 5e^{7-x}$
16. $2x - 1 = e^{\ln x^2}$
17. $9^x = 2e^{x^2}$

For Exercises 18–23, solve for t. Assume a and b are positive constants and k is nonzero.

18. $a = b^t$
19. $P = P_0 a^t$
20. $Q = Q_0 a^{nt}$
21. $P_0 a^t = Q_0 b^t$
22. $a = be^t$
23. $P = P_0 e^{kt}$

In Exercises 24–27, put the functions in the form $P = P_0 e^{kt}$.

24. $P = 15(1.5)^t$
25. $P = 10(1.7)^t$
26. $P = 174(0.9)^t$
27. $P = 4(0.55)^t$

Find the inverse function in Exercises 28–30.

28. $p(t) = (1.04)^t$
29. $f(t) = 50e^{0.1t}$
30. $f(t) = 1 + \ln t$

Problems

31. Without a calculator or computer, match the functions e^x, $\ln x$, x^2, and $x^{1/2}$ to their graphs in Figure 1.36.

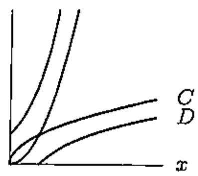


Figure 1.36

32. The population of a region is growing exponentially. There were 40,000,000 people in 1980 ($t = 0$) and 56,000,000 in 1990. Find an expression for the population at any time t, in years. What population would you predict for the year 2000? What is the doubling time?
33. What is the doubling time of prices which are increasing by 5% a year?
34. If the size of a bacteria colony doubles in 5 hours, how long will it take for the number of bacteria to triple?
35. One hundred kilograms of a radioactive substance decay to 40 kg in 10 years. How much remains after 20 years?
36. Find the half-life of a radioactive substance that is reduced by 30% in 20 hours.
37. The sales at Borders bookstores went from \$78 million in 1991 to \$412 million in 1994. Find an exponential function to model the sales as a function of years since 1991. What is the continuous percent growth rate, per year, of sales?
38. Owing to an innovative rural public health program, infant mortality in Senegal, West Africa, is being reduced at a rate of 10% per year. How long will it take for infant mortality to be reduced by 50%?
39. Find the equation of the line l in Figure 1.37.

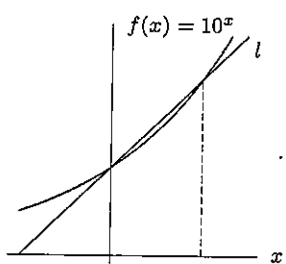


Figure 1.37

40. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A, in mg, remaining in the body is given by $A = 10(0.82)^t$.
    (a) What was the initial amount taken?
    (b) What percent of the drug leaves the body each hour?
    (c) How much of the drug is left in the body 6 hours after the dose is administered?
    (d) How long is it until only 1 mg of the drug remains in the body?
41. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.
    (a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
    (b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
    (c) Use logarithms to find the half-life of caffeine.
42. In 1980, there were about 170 million vehicles (cars and trucks) and about 227 million people in the United States. The number of vehicles has been growing at 4% a year, while the population has been growing at 1% a year. When was there, on average, one vehicle per person?
43. The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing according to the function $P = P_0e^{-kt}$, where t is time in hours. If 10% of the pollution is removed in the first five hours:
    (a) What percentage of the pollution is left after 10 hours?
    (b) How long is it before the pollution is reduced by 50%?
    (c) Plot a graph of pollution against time. Show the results of your calculations on the graph.
    (d) Explain why the quantity of pollutant might decrease in this way.
44. Air pressure, P, decreases exponentially with the height, h, in meters above sea level:

$$P = P_0e^{-0.00012h}$$

where P_0 is the air pressure at sea level.
    (a) At the top of Mount McKinley, height 6198 meters (about 20,330 feet), what is the air pressure, as a percent of the pressure at sea level?
    (b) The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?
45. The half-life of radioactive strontium-90 is 29 years. In 1960, radioactive strontium-90 was released into the atmosphere during testing of nuclear weapons, and was absorbed into people's bones. How many years does it take until only 10% of the original amount absorbed remains?
46. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.

1.5 TRIGONOMETRIC FUNCTIONS

Trigonometry originated as part of the study of triangles. The name *tri-gon-o-metry* means the measurement of three-cornered figures, and the first definitions of the trigonometric functions were in terms of triangles. However, the trigonometric functions can also be defined using the unit circle, a definition that makes them periodic, or repeating. Many naturally occurring processes are also periodic. The water level in a tidal basin, the blood pressure in a heart, an alternating current, and the position of the air molecules transmitting a musical note all fluctuate regularly. Such phenomena can be represented by trigonometric functions.

We use the three trigonometric functions found on a calculator: the sine, the cosine, and the tangent.

Radians

There are two commonly used ways to represent the input of the trigonometric functions: radians and degrees. The formulas of calculus, as you will see, are neater in radians than in degrees.

> An angle of 1 **radian** is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise. (See Figure 1.38(a).) A unit circle has radius 1.

An angle of 2 radians cuts off an arc of length 2 on a unit circle. A negative angle, such as $-1/2$ radians, cuts off an arc of length $1/2$, but measured clockwise. (See Figure 1.38(b).)

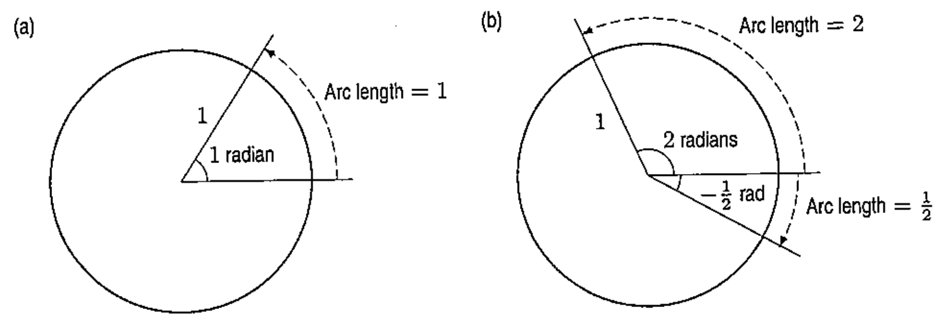


Figure 1.38: Radians defined using unit circle

It is useful to think of angles as rotations, since then we can make sense of angles larger than 360°; for example, an angle of 720° represents two complete rotations counterclockwise. Since one full rotation of 360° cuts off an arc of length 2π, the circumference of the unit circle, it follows that

$$360^\circ = 2\pi \text{ radians}, \quad \text{so} \quad 180^\circ = \pi \text{ radians}.$$

In other words, 1 radian $= 180^\circ/\pi$, so one radian is about 60°. The word radians is often dropped, so if an angle or rotation is referred to without units, it is understood to be in radians.

Radians are useful for computing the length of an arc in any circle. If the circle has radius r and the arc cuts off an angle θ, as in Figure 1.39, then we have the following relation:

> Arc length $= s = r\theta.$

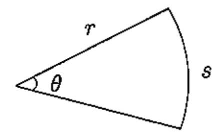


Figure 1.39: Arc length of a sector of a circle

The Sine and Cosine Functions

The two basic trigonometric functions—the sine and cosine—are defined using a unit circle. In Figure 1.40, an angle of t radians is measured counterclockwise around the circle from the point $(1,0)$. If P has coordinates (x, y), we define

$$\cos t = x \quad \text{and} \quad \sin t = y.$$

We assume that the angles are *always* in radians unless specified otherwise.

Since the equation of the unit circle is $x^2+y^2 = 1$, we have the following fundamental identity

$$\cos^2 t + \sin^2 t = 1.$$

As t increases and P moves around the circle, the values of $\sin t$ and $\cos t$ oscillate between 1 and -1, and eventually repeat as P moves through points where it has been before. If t is negative, the angle is measured clockwise around the circle.

Amplitude, Period, and Phase

The graphs of sine and cosine are shown in Figure 1.41. Notice that sine is an odd function, and cosine is even. The maximum and minimum values of sine and cosine are $+1$ and -1, because those are the maximum and minimum values of y and x on the unit circle. After the point P has moved around the complete circle once, the values of $\cos t$ and $\sin t$ start to repeat; we say the functions are *periodic.*

For any periodic function of time, the
- **Amplitude** is half the distance between the maximum and minimum values (if it exists).
- **Period** is the smallest time needed for the function to execute one complete cycle.

The amplitude of $\cos t$ and $\sin t$ is 1, and the period is 2π. Why 2π? Because that's the value of t when the point P has gone exactly once around the circle. (Remember that $360° = 2\pi$ radians.)

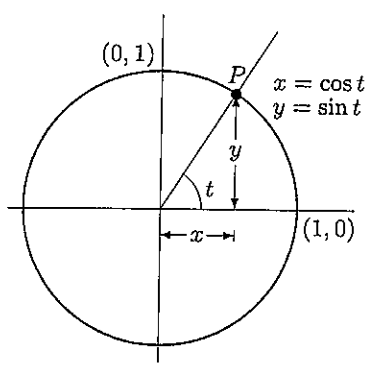


Figure 1.40: The definitions of $\sin t$ and $\cos t$

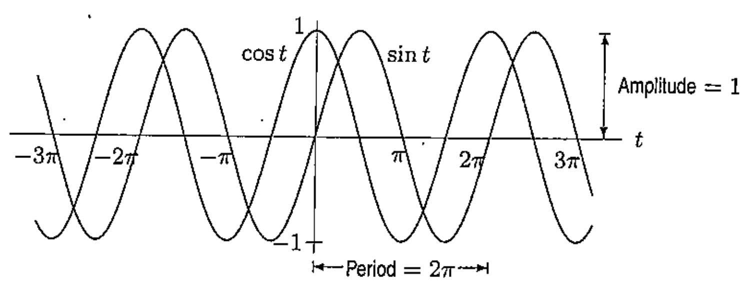


Figure 1.41: Graphs of $\cos t$ and $\sin t$

In Figure 1.41, we see that the sine and cosine graphs are exactly the same shape, only shifted horizontally. Since the cosine graph is the sine graph shifted $\pi/2$ to the left,

$$\cos t = \sin(t + \pi/2).$$

Equivalently, the sine graph is the cosine graph shifted $\pi/2$ to the right, so

$$\sin t = \cos(t - \pi/2).$$

We say that the *phase difference* or *phase shift*[5] between $\sin t$ and $\cos t$ is $\pi/2$.

Functions whose graphs are the shape of a sine or cosine curve are called *sinusoidal* functions.

> To describe arbitrary amplitudes and periods of sinusoidal functions, we use functions of the form
>
> $$f(t) = A\sin(Bt) \qquad \text{and} \qquad g(t) = A\cos(Bt),$$
>
> where $|A|$ is the amplitude and $2\pi/|B|$ is the period.
>
> The graph of a sinusoidal function is shifted horizontally by a distance $|h|$ when t is replaced by $t - h$ or $t + h$.
>
> Functions of the form $f(t) = A\sin(Bt) + C$ and $g(t) = A\cos(Bt) + C$ have graphs which are shifted vertically and oscillate about the value C.

Example 1 Find and show on a graph the amplitude and period of the functions

(a) $y = 5\sin(2t)$ (b) $y = -5\sin\left(\frac{t}{2}\right)$ (c) $y = 1 + 2\sin t$

Solution

(a) From Figure 1.42, you can see that the amplitude of $y = 5\sin(2t)$ is 5 because the factor of 5 stretches the oscillations up to 5 and down to -5. The period of $y = \sin(2t)$ is π, because when t changes from 0 to π, the quantity $2t$ changes from 0 to 2π, so the sine function goes through one complete oscillation.

(b) Figure 1.43 shows that the amplitude of $y = -5\sin(t/2)$ is again 5, because the negative sign reflects the oscillations in the t-axis, but does not change how far up or down they go. The period of $y = -5\sin(t/2)$ is 4π because when t changes from 0 to 4π, the quantity $t/2$ changes from 0 to 2π, so the sine function goes through one complete oscillation.

(c) The 1 shifts the graph $y = 2\sin t$ up by 1. Since $y = 2\sin t$ has an amplitude of 2 and a period of 2π, the graph of $y = 1 + 2\sin t$ goes up to 3 and down to -1, and has a period of 2π. (See Figure 1.44.) Thus, $y = 1 + 2\sin t$ also has amplitude 2.

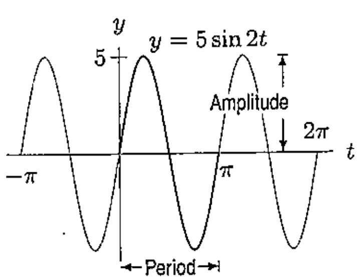


Figure 1.42: Amplitude = 5, period = π

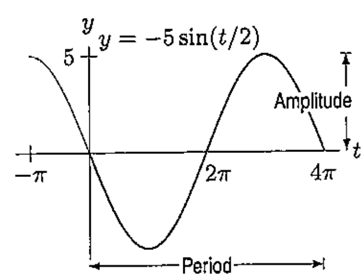


Figure 1.43: Amplitude = 5, period = 4π

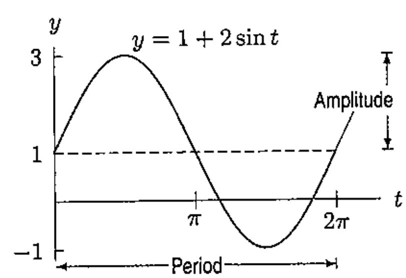
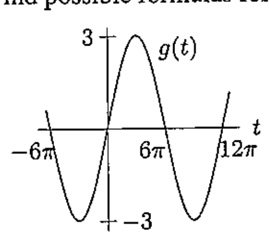


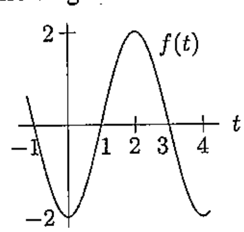
Figure 1.44: Amplitude = 2, period = 2π

Example 2 Find possible formulas for the following sinusoidal functions.

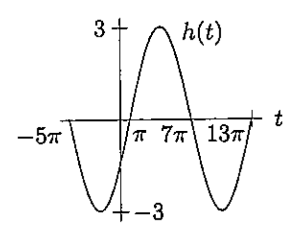
(a)



(b)



(c)



[5] Phase shift is defined in Section 11.10 on page 585.

Solution

(a) This function looks like a sine function of amplitude 3, so $g(t) = 3\sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 12\pi$, when t changes by 12π, the quantity Bt changes by 2π. This means $B \cdot 12\pi = 2\pi$, so $B = 1/6$. Therefore, $g(t) = 3\sin(t/6)$ has the graph shown.

(b) This function looks like an upside down cosine function with amplitude 2, so $f(t) = -2\cos(Bt)$. The function completes one oscillation between $t = 0$ and $t = 4$. Thus, when t changes by 4, the quantity Bt changes by 2π, so $B \cdot 4 = 2\pi$, or $B = \pi/2$. Therefore, $f(t) = -2\cos(\pi t/2)$ has the graph shown.

(c) This function looks like the function $g(t)$ in part (a), but shifted a distance of π to the right. Since $g(t) = 3\sin(t/6)$, we replace t by $(t - \pi)$ to obtain $h(t) = 3\sin[(t - \pi)/6]$.

Example 3 On February 10, 1990, high tide in Boston was at midnight. The water level at high tide was 9.9 feet; later, at low tide, it was 0.1 feet. Assuming the next high tide is at exactly 12 noon and that the height of the water is given by a sine or cosine curve, find a formula for the water level in Boston as a function of time.

Solution

Let y be the water level in feet, and let t be the time measured in hours from midnight. The oscillations have amplitude 4.9 feet $(= (9.9 - 0.1)/2)$ and period 12, so $12B = 2\pi$ and $B = \pi/6$. Since the water is highest at midnight, when $t = 0$, the oscillations are best represented by a cosine function. (See Figure 1.45.) We can say

$$\text{Height above average} = 4.9\cos\left(\frac{\pi}{6}t\right).$$

Since the average water level was 5 feet $(= (9.9 + 0.1)/2)$, we shift the cosine up by adding 5:

$$y = 5 + 4.9\cos\left(\frac{\pi}{6}t\right).$$

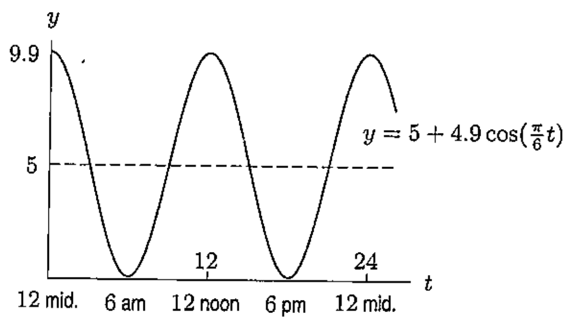


Figure 1.45: Function approximating the tide in Boston on February 10, 1990

Example 4 Of course, there's something wrong with the assumption in Example 3 that the next high tide is at noon. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as in fact it does. The interval between successive high tides actually averages about 12 hours 24 minutes. Using this, give a more accurate formula for the height of the water as a function of time.

Solution

The period is 12 hours 24 minutes = 12.4 hours, so $B = 2\pi/12.4$, giving

$$y = 5 + 4.9\cos\left(\frac{2\pi}{12.4}t\right) = 5 + 4.9\cos(0.507t).$$

Example 5 Use the information from Example 4 to write a formula for the water level in Boston on a day when the high tide is at 2 pm.

Solution When the high tide is at midnight

$$y = 5 + 4.9\cos(0.507t).$$

Since 2 pm is 14 hours after midnight, we replace t by $(t - 14)$. Therefore, on a day when the high tide is at 2 pm,

$$y = 5 + 4.9\cos(0.507(t - 14)).$$

The Tangent Function

If t is any number with $\cos t \neq 0$, we define the tangent function as follows

$$\tan t = \frac{\sin t}{\cos t}.$$

Figure 1.40 on page 30 shows the geometrical meaning of the tangent function: $\tan t$ is the slope of the line through the origin $(0, 0)$ and the point $P = (\cos t, \sin t)$ on the unit circle.

The tangent function is undefined wherever $\cos t = 0$, namely, at $t = \pm\pi/2, \pm 3\pi/2, \ldots$, and it has a vertical asymptote at each of these points. The function $\tan t$ is positive where $\sin t$ and $\cos t$ have the same sign. The graph of the tangent is shown in Figure 1.46.

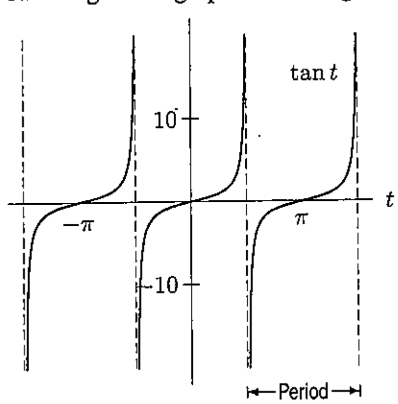


Figure 1.46: The tangent function

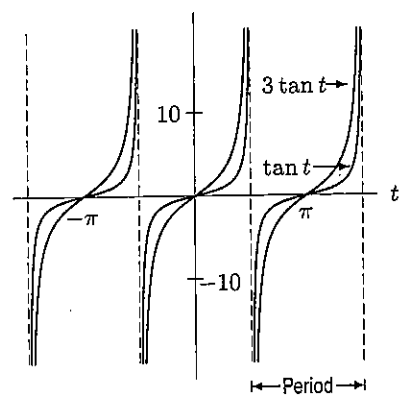


Figure 1.47: Multiple of tangent

The tangent function has period π, because it repeats every π units. Does it make sense to talk about the amplitude of the tangent function? Not if we're thinking of the amplitude as a measure of the size of the oscillation, because the tangent becomes infinitely large near each vertical asymptote. We can still multiply the tangent by a constant, but that constant no longer represents an amplitude. (See Figure 1.47.)

The Inverse Trigonometric Functions

On occasion, you may need to find a number with a given sine. For example, you might want to find x such that

$$\sin x = 0$$

or such that

$$\sin x = 0.3.$$

The first of these equations has solutions $x = 0, \pm\pi, \pm 2\pi, \ldots$. The second equation also has infinitely many solutions. Using a calculator, we get

$$x \approx 0.305, 2.84, 0.305 \pm 2\pi, 2.84 \pm 2\pi, \ldots.$$

For each equation, we pick out the solution between $-\pi/2$ and $\pi/2$ as the preferred solution. For example, the preferred solution to $\sin x = 0$ is $x = 0$, and the preferred solution to $\sin x = 0.3$ is $x = 0.305$. We define the inverse sine, written "arcsin" or "$\sin^{-1}$," as the function which gives the preferred solution.

> For $-1 \le y \le 1$,
> $$\arcsin y = x$$
> means
> $$\sin x = y \quad \text{with} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Thus the arcsine is the inverse function to the piece of the sine function having domain $[-\pi/2, \pi/2]$. (See Table 1.14 and Figure 1.48.) On a calculator, the arcsine function[6] is usually denoted by $\boxed{\sin^{-1}}$.

Table 1.14 *Values of* $\sin x$ *and* $\sin^{-1} x$

x	$\sin x$
$-\frac{\pi}{2}$	-1.000
-1.0	-0.841
-0.5	-0.479
0.0	0.000
0.5	0.479
1.0	0.841
$\frac{\pi}{2}$	1.000

x	$\sin^{-1} x$
-1.000	$-\frac{\pi}{2}$
-0.841	-1.0
-0.479	-0.5
0.000	0.0
0.479	0.5
0.841	1.0
1.000	$\frac{\pi}{2}$

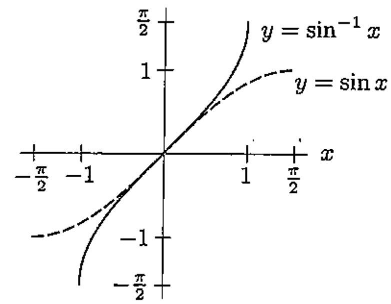


Figure 1.48: The arcsine function

The inverse tangent, written "arctan" or "$\tan^{-1}$," is the inverse function for the piece of the tangent function having the domain $-\pi/2 < x < \pi/2$. On a calculator, the inverse tangent is usually denoted by $\boxed{\tan^{-1}}$. The graph of the arctangent is shown in Figure 1.50.

> For any y,
> $$\arctan y = x$$
> means
> $$\tan x = y \quad \text{with} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The inverse cosine function, written "arccos" or "$\cos^{-1}$," is discussed in Problem 47. The range of the arccosine function is $0 \le x \le \pi$.

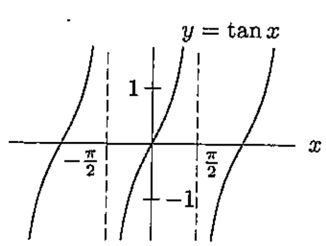


Figure 1.49: The tangent function

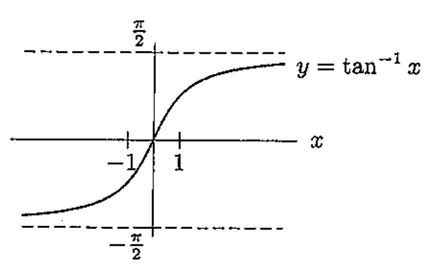


Figure 1.50: The arctangent function

[6]Note that $\sin^{-1} x = \arcsin x$ is not the same as $(\sin x)^{-1} = 1/\sin x$.

Exercises and Problems for Section 1.5

Exercises

For Exercises 1–9, draw the angle using a ray through the origin, and determine whether the sine, cosine, and tangent of that angle are positive, negative, zero, or undefined.

1. $\frac{3\pi}{2}$ **2.** 2π **3.** $\frac{\pi}{4}$

4. 3π **5.** $\frac{\pi}{6}$ **6.** $\frac{4\pi}{3}$

7. $\frac{-4\pi}{3}$ **8.** 4 **9.** -1

Given that $\sin(\pi/12) = 0.259$ and $\cos(\pi/5) = 0.809$, compute (without using the trigonometric functions on your calculator) the quantities in Exercises 10–12. You may want to draw a picture showing the angles involved, and then check your answers on a calculator.

10. $\cos\left(-\frac{\pi}{5}\right)$ **11.** $\sin\frac{\pi}{5}$ **12.** $\cos\frac{\pi}{12}$

13. Consider the function $y = 5 + \cos(3x)$.

(a) What is its amplitude?
(b) What is its period?
(c) Sketch its graph.

Find the period and amplitude in Exercises 14–17.

14. $y = 7\sin(3t)$ **15.** $z = 3\cos(u/4) + 5$

16. $w = 8 - 4\sin(2x + \pi)$ **17.** $r = 0.1\sin(\pi t) + 2$

18. Without a calculator or computer, match the formulas with the graphs in Figure 1.51.

(a) $y = 2\cos(t - \pi/2)$ (b) $y = 2\cos t$
(c) $y = 2\cos(t + \pi/2)$

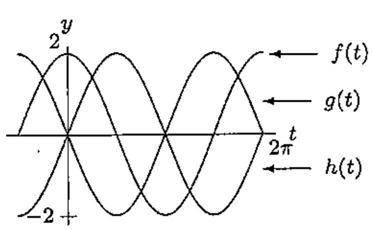


Figure 1.51

For Exercises 19–28, find a possible formula for each graph.

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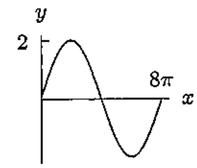

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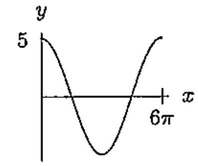

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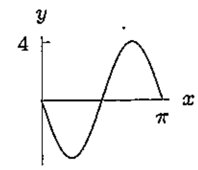

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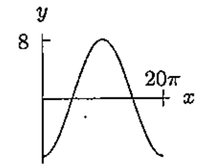

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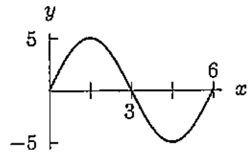

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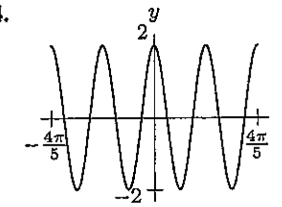

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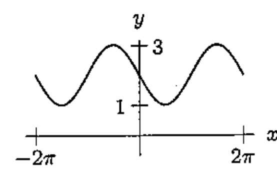

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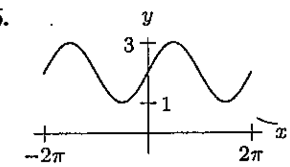

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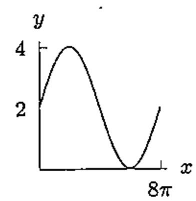

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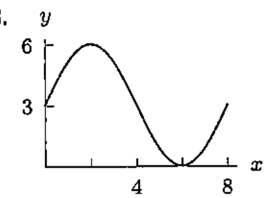

In Exercises 29–33, find a solution to the equation if possible. Give the answer in exact form and in decimal form.

29. $2 = 5\sin(3x)$ **30.** $1 = 8\cos(2x + 1) - 3$

31. $8 = 4\tan(5x)$ **32.** $1 = 8\tan(2x + 1) - 3$

33. $8 = 4\sin(5x)$

Problems

34. A compact disk spins at a rate of 200 to 500 revolutions per minute. What are the equivalent rates measured in radians per second?

35. When a car's engine makes less than about 200 revolutions per minute, it stalls. What is the period of the rotation of the engine when it is about to stall?

36. What is the difference between $\sin x^2$, $\sin^2 x$, and $\sin(\sin x)$? Express each of the three as a composition. (Note: $\sin^2 x$ is another way of writing $(\sin x)^2$.)

37. On the graph of $y = \sin x$, points P and Q are at consecutive lowest and highest points. Find the slope of the line through P and Q.

38. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t, in hours since midnight by

$$y = D + A\cos(B(t - C)).$$

(a) What is the physical meaning of D?
(b) What is the value of A?
(c) What is the value of B? Assume the time between successive high tides is 12.4 hours.
(d) What is the physical meaning of C?

39. A mass is oscillating on the end of a spring. The distance, y, of the mass from its equilibrium position is given by

$$y = y_0 \cos(2\pi\omega t).$$

Here y is in centimeters, t is time in seconds, and y_0 and ω are positive constants.

(a) What is the meaning of y_0 in terms of oscillations?
(b) How many oscillations are completed in 1 second?

40. In an electrical outlet, the voltage, V, in volts, is given as a function of time, t, in seconds, by the formula

$$V = V_0 \sin(120\pi t).$$

(a) What does V_0 represent in terms of voltage?
(b) What is the period of this function?
(c) How many oscillations are completed in 1 second?

41. In a US household, the voltage in volts in an electric outlet is given by

$$V = 156\sin(120\pi t),$$

where t is in seconds. However, in a European house, the voltage is given (in the same units) by

$$V = 339\sin(100\pi t).$$

Compare the voltages in the two regions, considering the maximum voltage and number of cycles (oscillations) per second.

42. A baseball hit at an angle of θ to the horizontal with initial velocity v_0 has horizontal range, R, given by

$$R = \frac{v_0^2}{g}\sin(2\theta).$$

Here g is the acceleration due to gravity. Sketch R as a function of θ for $0 \le \theta \le \pi/2$. What angle gives the maximum range? What is the maximum range?

43. A population of animals oscillates sinusoidally between a low of 700 on January 1 and a high of 900 on July 1.

(a) Graph the population against time.
(b) Find a formula for the population as a function of time, t, in months since the start of the year.

44. The desert temperature, H, oscillates daily between 40° F at 5 am and 80° F at 5 pm. Write a possible formula for H in terms of t, measured in hours from 5 am.

45. The point P is rotating around a circle of radius 5 shown in Figure 1.52. The angle θ, in radians, is given as a function of time, t, by the graph in Figure 1.53.

(a) Estimate the coordinates of P when $t = 1.5$.
(b) Describe in words the motion of the point P on the circle.

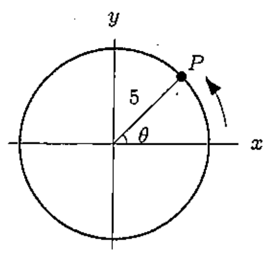

Figure 1.52

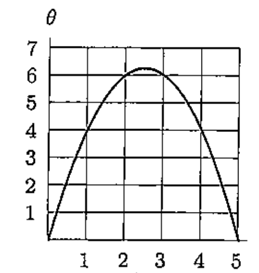

Figure 1.53

46. Find the area of the trapezoidal cross-section of the irrigation canal shown in Figure 1.54.

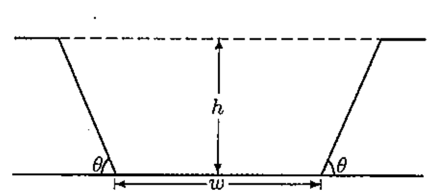

Figure 1.54

47. This problem introduces the arccosine function, or inverse cosine, denoted by $\boxed{\cos^{-1}}$ on most calculators.

(a) Using a calculator set in radians, make a table of values, to two decimal places, of $g(x) = \arccos x$, for $x = -1, -0.8, -0.6, \ldots, 0, \ldots, 0.6, 0.8, 1$.
(b) Sketch the graph of $g(x) = \arccos x$.
(c) Why is the domain of the arccosine the same as the domain of the arcsine?
(d) Why is the range of the arccosine *not* the same as the range of the arcsine? To answer this, look at how the domain of the original sine function was restricted to construct the arcsine. Why can't the domain of the cosine be restricted in exactly the same way to construct the arccosine?

1.6 POWERS, POLYNOMIALS, AND RATIONAL FUNCTIONS

Power Functions

A *power function* is a function in which the dependent variable is proportional to a power of the independent variable:

> A **power function** has the form
>
> $f(x) = kx^p$, where k and p are constant.

For example, the volume, V, of a sphere of radius r is given by

$$V = g(r) = \frac{4}{3}\pi r^3.$$

As another example, the gravitational force, F, on a unit mass at a distance r from the center of the earth is given by Newton's Law of Gravitation, which says that, for some positive constant k,

$$F = \frac{k}{r^2} \quad \text{or} \quad F = kr^{-2}.$$

We consider the graphs of the power functions x^n, with n a positive integer. Figures 1.55 and 1.56 show that the graphs fall into two groups: odd and even powers. For n greater than 1, the odd powers have a "seat" at the origin and are increasing everywhere else. The even powers are first decreasing and then increasing. For large x, the higher the power of x, the faster the function climbs.

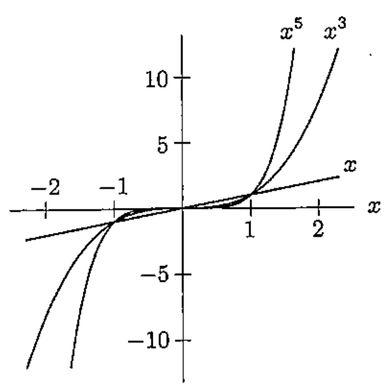

Figure 1.55: Odd powers of x: "Seat" shaped for $n > 1$

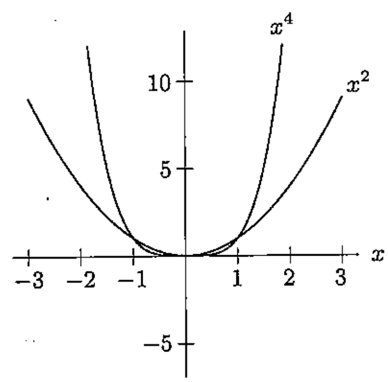

Figure 1.56: Even powers of x: ⋃-shaped

Exponentials and Power Functions: Which Dominate?

In everyday language, the word exponential is often used to imply very fast growth. But do exponential functions always grow faster than power functions? To determine what happens "in the long run," we often want to know which functions *dominate* as x gets arbitrarily large.

Let's consider $y = 2^x$ and $y = x^3$. The close-up view in Figure 1.57(a) shows that between $x = 2$ and $x = 4$, the graph of $y = 2^x$ lies below the graph of $y = x^3$. The far-away view in Figure 1.57(b) shows that the exponential function $y = 2^x$ eventually overtakes $y = x^3$. Figure 1.57(c), which gives a very far–away view, shows that, for large x, the value of x^3 is insignificant compared to 2^x. Indeed, 2^x is growing so much faster than x^3 that the graph of 2^x appears almost vertical in comparison to the more leisurely climb of x^3.

We say that Figure 1.57(a) gives a *local* view of the functions' behavior, whereas Figure 1.57(c) gives a *global* view.

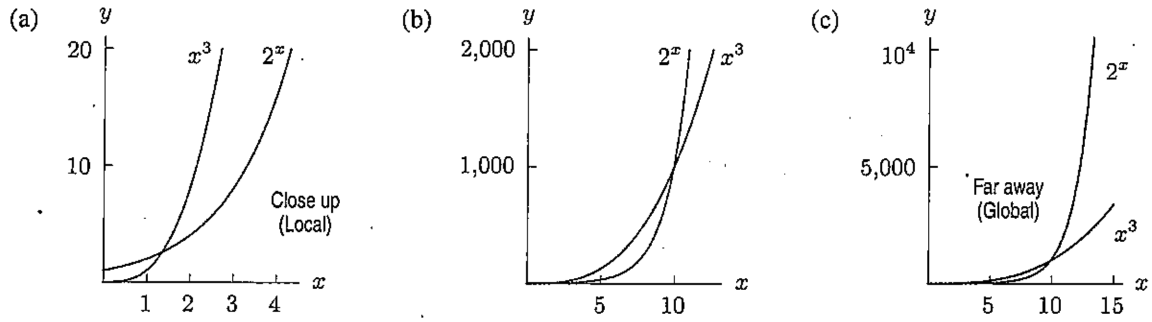

Figure 1.57: Comparison of $y = 2^x$ and $y = x^3$: Notice that $y = 2^x$ eventually dominates $y = x^3$

In fact, *every* exponential growth function eventually dominates *every* power function. Although an exponential function may be below a power function for some values of x, if we look at large enough x-values, a^x (with $a > 1$) will eventually dominate x^n, no matter what n is.

Polynomials

Polynomials are the sums of power functions with nonnegative integer exponents:

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Here n is a nonnegative integer called the *degree* of the polynomial, and $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, with leading coefficient $a_n \neq 0$. An example of a polynomial of degree $n = 3$ is

$$y = p(x) = 2x^3 - x^2 - 5x - 7.$$

In this case $a_3 = 2, a_2 = -1, a_1 = -5$, and $a_0 = -7$. The shape of the graph of a polynomial depends on its degree; typical graphs are shown in Figure 1.58. These graphs correspond to a positive coefficient for x^n; a negative leading coefficient turns the graph upside down. Notice that the quadratic "turns around" once, the cubic "turns around" twice, and the quartic (fourth degree) "turns around" three times. An n^{th} degree polynomial "turns around" at most $n - 1$ times (where n is a positive integer), but there may be fewer turns.

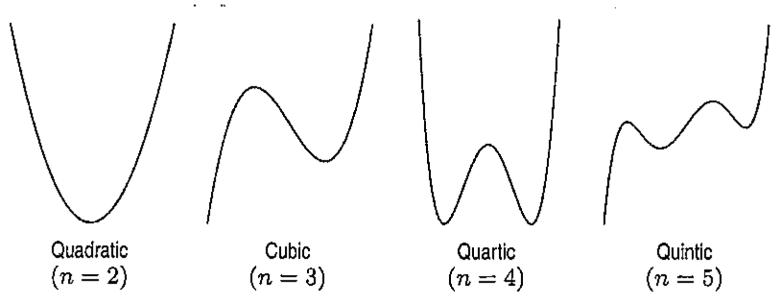

Figure 1.58: Graphs of typical polynomials of degree n

Example 1 Find possible formulas for the polynomials whose graphs are in Figure 1.59.

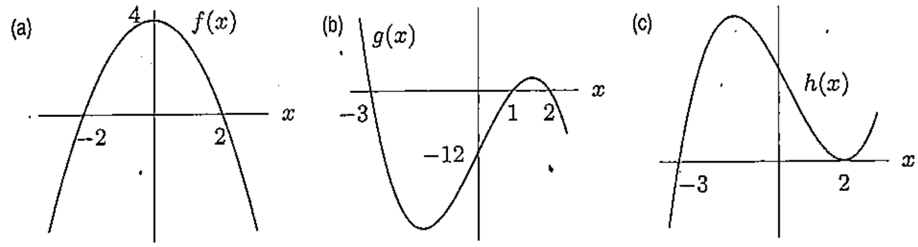

Figure 1.59: Graphs of polynomials

Solution

(a) This graph appears to be a parabola, turned upside down, and moved up by 4, so
$$f(x) = -x^2 + 4.$$
The negative sign turns the parabola upside down and the $+4$ moves it up by 4. Notice that this formula does give the correct x-intercepts since $0 = -x^2 + 4$ has solutions $x = \pm 2$. These values of x are called *zeros* of f.

We can also solve this problem by looking at the x-intercepts first, which tell us that $f(x)$ has factors of $(x + 2)$ and $(x - 2)$. So
$$f(x) = k(x + 2)(x - 2).$$
To find k, use the fact that the graph has a y-intercept of 4, so $f(0) = 4$, giving
$$4 = k(0 + 2)(0 - 2),$$
or $k = -1$. Therefore, $f(x) = -(x + 2)(x - 2)$, which multiplies out to $-x^2 + 4$.

Note that $f(x) = 4 - x^4/4$ also has the same basic shape, but is flatter near $x = 0$. There are many possible answers to these questions.

(b) This looks like a cubic with factors $(x + 3)$, $(x - 1)$, and $(x - 2)$, one for each intercept:
$$g(x) = k(x + 3)(x - 1)(x - 2).$$
Since the y-intercept is -12, we have
$$-12 = k(0 + 3)(0 - 1)(0 - 2).$$
So $k = -2$, and we get the cubic polynomial
$$g(x) = -2(x + 3)(x - 1)(x - 2).$$

(c) This also looks like a cubic with zeros at $x = 2$ and $x = -3$. Notice that at $x = 2$ the graph of $h(x)$ touches the x-axis but does not cross it, whereas at $x = -3$ the graph crosses the x-axis. We say that $x = 2$ is a *double zero*, but that $x = -3$ is a single zero.

To find a formula for $h(x)$, imagine the graph of $h(x)$ to be slightly lower down, so that the graph has one x-intercept near $x = -3$ and two near $x = 2$, say at $x = 1.9$ and $x = 2.1$. Then a formula would be
$$h(x) \approx k(x + 3)(x - 1.9)(x - 2.1).$$
Now move the graph back to its original position. The zeros at $x = 1.9$ and $x = 2.1$ move toward $x = 2$, giving
$$h(x) = k(x + 3)(x - 2)(x - 2) = k(x + 3)(x - 2)^2.$$
The double zero leads to a repeated factor, $(x - 2)^2$. Notice that when $x > 2$, the factor $(x - 2)^2$ is positive, and when $x < 2$, the factor $(x - 2)^2$ is still positive. This reflects the fact that $h(x)$ does not change sign near $x = 2$. Compare this with the behavior near the single zero at $x = -3$, where h does change sign.

We cannot find k, as no coordinates are given for points off of the x-axis. Any positive value of k stretches the graph vertically but does not change the zeros, so any positive k works.

Example 2 Using a calculator or computer, graph $y = x^4$ and $y = x^4 - 15x^2 - 15x$ for $-4 \le x \le 4$ and for $-20 \le x \le 20$. Set the y range to $-100 \le y \le 100$ for the first domain, and to $-100 \le y \le 200{,}000$ for the second. What do you observe?

Solution From the graphs in Figure 1.60 we see that close up ($-4 \le x \le 4$) the graphs look different; from far away, however, they are almost indistinguishable. The reason is that the leading terms (those with the highest power of x) are the same, namely x^4, and for large values of x, the leading term dominates the other terms.

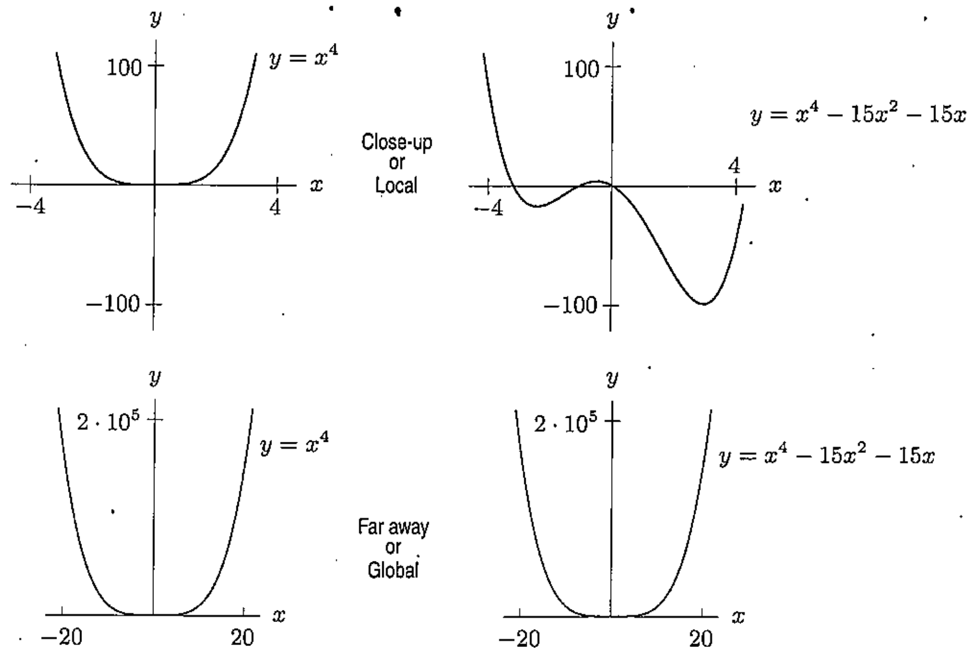

Figure 1.60: Local and global views of $y = x^4$ and $y = x^4 - 15x^2 - 15x$

Rational Functions

Rational functions are ratios of polynomials, p and q:

$$f(x) = \frac{p(x)}{q(x)}.$$

Example 3 Look at a graph and explain the behavior of $y = \dfrac{1}{x^2 + 4}$.

Solution The function is even, so the graph is symmetric about the y-axis. As x gets larger, the denominator gets larger, making the value of the function closer to 0. Thus the graph gets arbitrarily close to the x-axis as x increases without bound. See Figure 1.61.

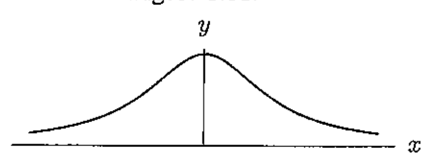

Figure 1.61: Graph of $y = \frac{1}{x^2+4}$

In the previous example, we say that $y = 0$ (i.e. the x-axis) is a *horizontal asymptote.* Writing "$\rightarrow$" to mean "tends to," we have $y \rightarrow 0$ as $x \rightarrow \infty$ and $y \rightarrow 0$ as $x \rightarrow -\infty$.

> If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$, then the line $y = L$ is called a **horizontal asymptote.**[7] This occurs when
>
> $$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow \infty \qquad \text{or} \qquad f(x) \rightarrow L \quad \text{as} \quad x \rightarrow -\infty.$$
>
> If the graph of $y = f(x)$ approaches the vertical line $x = K$ as $x \rightarrow K$ from one side or the other, that is, if
>
> $$y \rightarrow \infty \quad \text{or} \quad y \rightarrow -\infty \quad \text{when} \quad x \rightarrow K,$$
>
> then the line $x = K$ is called a **vertical asymptote.**

The graphs of rational functions may have vertical asymptotes where the denominator is zero. For example, the function in Example 3 has no vertical asymptotes as the denominator is never zero. The function in Example 4 has two vertical asymptotes corresponding to the two zeros in the denominator.

Rational functions have horizontal asymptotes if $f(x)$ approaches a finite number as $x \rightarrow \infty$ or $x \rightarrow -\infty$. We call the behavior of a function as $x \rightarrow \pm\infty$ its *end behavior.*

Example 4 Look at a graph and explain the behavior of $y = \dfrac{3x^2 - 12}{x^2 - 1}$, including end behavior.

Solution Factoring gives

$$y = \frac{3x^2 - 12}{x^2 - 1} = \frac{3(x+2)(x-2)}{(x+1)(x-1)}$$

so $x = \pm 1$ are vertical asymptotes. If $y = 0$, then $3(x+2)(x-2) = 0$ or $x = \pm 2$; these are the x-intercepts. Note that zeros of the denominator give rise to the vertical asymptotes, whereas zeros of the numerator give rise to x-intercepts. Substituting $x = 0$ gives $y = 12$; this is the y-intercept. The function is even, so the graph is symmetric about the y-axis.

To see what happens as $x \rightarrow \pm\infty$, look at the y-values in Table 1.15. Clearly y is getting closer to 3 as x gets large positively or negatively. Alternatively, realize that as $x \rightarrow \pm\infty$, only the highest powers of x matter. For large x, the 12 and the 1 are insignificant compared to x^2, so

$$y = \frac{3x^2 - 12}{x^2 - 1} \approx \frac{3x^2}{x^2} = 3 \quad \text{for large } x.$$

So $y \rightarrow 3$ as $x \rightarrow \pm\infty$, and therefore the horizontal asymptote is $y = 3$. See Figure 1.62. Since, for $x > 1$, the denominator is positive and the numerator is less than the denominator, the graph lies *below* its asymptote. (Why doesn't the graph lie below $y = 3$ when $-1 < x < 1$?)

Table 1.15 *Values of* $y = \frac{3x^2-12}{x^2-1}$

x	$y = \frac{3x^2-12}{x^2-1}$
± 10	2.909091
± 100	2.999100
± 1000	2.999991

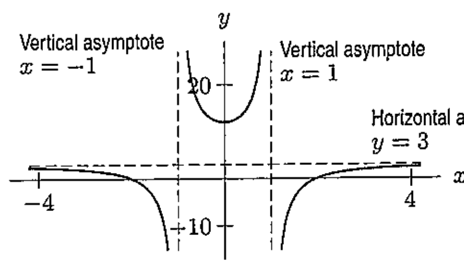

Figure 1.62: Graph of the function $y = \frac{3x^2-12}{x^2-1}$

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[7]We are assuming that $f(x)$ gets arbitrarily close to L as $x \rightarrow \infty$.

Exercises and Problems for Section 1.6

Exercises

In Exercises 1–2, which function dominates as $x \to \infty$?

1. $10 \cdot 2^x$ or $72{,}000x^{12}$ **2.** $0.25\sqrt{x}$ or $25{,}000x^{-3}$

For Problems 3–4, what happens to the value of the function as $x \to \infty$ and as $x \to -\infty$?

3. $y = 0.25x^3 + 3$ **4.** $y = 2 \cdot 10^{4x}$

5. Each of the graphs in Figure 1.63 is of a polynomial. The windows are large enough to show global behavior.

(a) What is the minimum possible degree of the polynomial?
(b) Is the leading coefficient of the polynomial positive or negative?

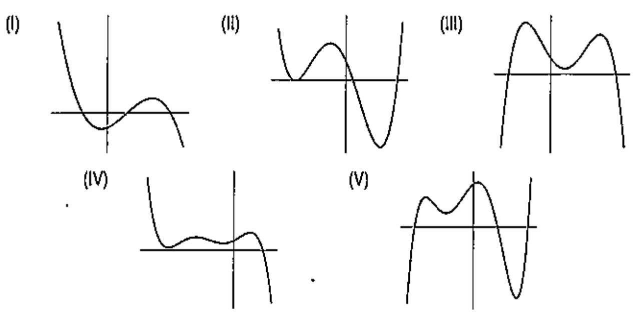


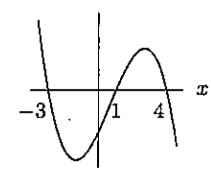
Figure 1.63

6. For each function, fill in the blanks in the statements:
$f(x) \to$ __________ as $x \to -\infty$,
$f(x) \to$ __________ as $x \to +\infty$.

(a) $f(x) = 17 + 5x^2 - 12x^3 - 5x^4$
(b) $f(x) = \dfrac{3x^2 - 5x + 2}{2x^2 - 8}$
(c) $f(x) = e^x$

Find possible formulas for the graphs in Exercises 7–10.

7.



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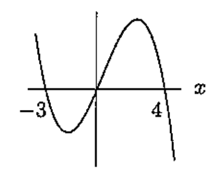

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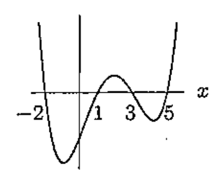

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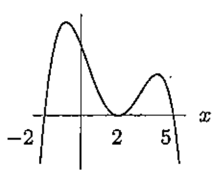


11. Use Figure 1.64 to graph each of the following. Label any intercepts or asymptotes that can be determined.

(a) $y = f(x) + 3$ (b) $y = 2f(x)$
(c) $y = f(x + 4)$ (d) $y = 4 - f(x)$

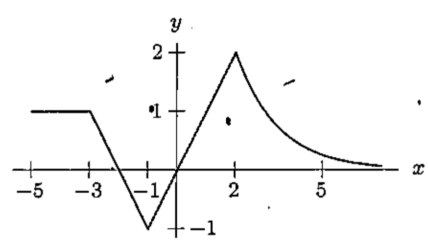
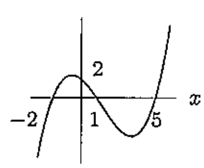


Figure 1.64

Problems

Find cubic polynomials for the graphs in Exercises 12–13.

12.



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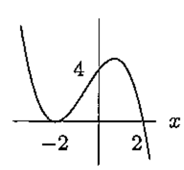

14. Which of the functions I–III meet each of the following descriptions? There may be more than one function for each description, or none at all.

(a) Horizontal asymptote of $y = 1$.
(b) The x-axis is a horizontal asymptote.
(c) Symmetric about the y-axis.
(d) An odd function.
(e) Vertical asymptotes at $x = \pm 1$.

I. $y = \dfrac{x - 1}{x^2 + 1}$ II. $y = \dfrac{x^2 - 1}{x^2 + 1}$ III. $y = \dfrac{x^2 + 1}{x^2 - 1}$

15. The DuBois formula relates a person's surface area, s, in m^2, to weight w, in kg, and height h, in cm, by

$$s = 0.01w^{0.25}h^{0.75}.$$

(a) What is the surface area of a person who weighs 65 kg and is 160 cm tall?
(b) What is the weight of a person whose height is 180 cm and who has a surface area of 1.5 m^2?
(c) For people of fixed weight 70 kg, solve for h as a function of s. Simplify your answer.

16. A box of fixed volume V has a square base with side length x. Write a formula for the height, h, of the box in terms of x and V. Sketch a graph of h versus x.

17. According to *Car and Driver*, an Alfa Romeo going at 70 mph requires 177 feet to stop. Assuming that the stopping distance is proportional to the square of velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph (its top speed).

18. Water is flowing down a cylindrical pipe of radius r.
   (a) Write a formula for the volume, V, of water that emerges from the end of the pipe in one second if the water is flowing at a rate of
      (i) 3 cm/sec (ii) k cm/sec
   (b) Graph your answer to part (a)(ii) as a function of
      (i) r, assuming k is constant
      (ii) k, assuming r is constant

19. Poiseuille's Law gives the rate of flow, R, of a gas through a cylindrical pipe in terms of the radius of the pipe, r, for a fixed drop in pressure between the two ends of the pipe.
   (a) Find a formula for Poiseuille's Law, given that the rate of flow is proportional to the fourth power of the radius.
   (b) If $R = 400$ cm^3/sec in a pipe of radius 3 cm for a certain gas, find a formula for the rate of flow of that gas through a pipe of radius r cm.
   (c) What is the rate of flow of the same gas through a pipe with a 5 cm radius?

20. The height of an object above the ground at time t is given by
$$s = v_0t - \frac{g}{2}t^2,$$
where v_0 is the initial velocity and g is the acceleration due to gravity.
   (a) At what height is the object initially?
   (b) How long is the object in the air before it hits the ground?
   (c) When will the object reach its maximum height?
   (d) What is that maximum height?

21. A pomegranate is thrown from ground level straight up into the air at time $t = 0$ with velocity 64 feet per second. Its height at time t seconds is $f(t) = -16t^2 + 64t$. Find the time it hits the ground and the time it reaches its highest point. What is the maximum height?

22. (a) If $f(x) = ax^2 + bx + c$, what can you say about the values of a, b, and c if:
      (i) $(1, 1)$ is on the graph of $f(x)$?
      (ii) $(1, 1)$ is the vertex of the graph of $f(x)$? [Hint: The axis of symmetry is $x = -b/(2a)$.]
      (iii) The y intercept of the graph is $(0, 6)$?
   (b) Find a quadratic function satisfying all three conditions.

23. Values of three functions are given in Table 1.16, rounded to two decimal places. One function is of the form $y = ab^t$, one is of the form $y = ct^2$, and one is of the form $y = kt^3$. Which function is which?

Table 1.16

t	$f(t)$	t	$g(t)$	t	$h(t)$
2.0	4.40	1.0	3.00	0.0	2.04
2.2	5.32	1.2	5.18	1.0	3.06
2.4	6.34	1.4	8.23	2.0	4.59
2.6	7.44	1.6	12.29	3.0	6.89
2.8	8.62	1.8	17.50	4.0	10.33
3.0	9.90	2.0	24.00	5.0	15.49

24. Values of three functions are given in Table 1.17, rounded to two decimal places. Two are power functions and one is an exponential. One of the power functions is a quadratic and one a cubic. Which one is exponential? Which one is quadratic? Which one is cubic?

Table 1.17

x	$f(x)$	x	$g(x)$	x	$k(x)$
8.4	5.93	5.0	3.12	0.6	3.24
9.0	7.29	5.5	3.74	1.0	9.01
9.6	8.85	6.0	4.49	1.4	17.66
10.2	10.61	6.5	5.39	1.8	29.19
10.8	12.60	7.0	6.47	2.2	43.61
11.4	14.82	7.5	7.76	2.6	60.91

25. A cubic polynomial with positive leading coefficient is shown in Figure 1.65 for $-10 \le x \le 10$ and $-10 \le y \le 10$. What can be concluded about the total number of zeros of this function? What can you say about the location of each of the zeros? Explain.

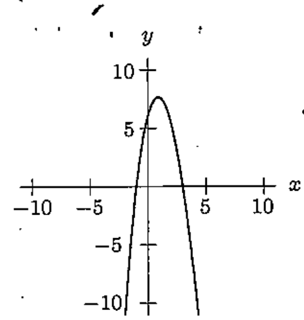

Figure 1.65

26. How many distinct roots can a polynomial of degree 5 have? (List all possibilities.) Sketch a possible graph for each case.

27. Suppose $f(x) = (x + a_1)^2$ and $g(x) = (x + a_2)^2$. Suppose $c > 0$ and $a_1 = -2a_2 = 2c$.
   (a) Solve $f(x) = g(x)$. Your answer should involve c but no other constants.
   (b) Sketch $f(x)$ and $g(x)$ on the same axes, labeling intercepts.
   (c) As c increases, what happens to the x- and y-coordinates of the point of intersection?

28. A rational function $y = f(x)$ is graphed in Figure 1.66. If $f(x) = g(x)/h(x)$ with $g(x)$ and $h(x)$ both quadratic functions, give possible formulas for $g(x)$ and $h(x)$.

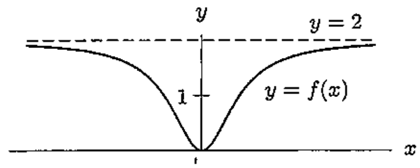

Figure 1.66

29. Match the following functions with the graphs in Figure 1.67. Assume $0 < b < a$.

(a) $y = \dfrac{a}{x} - x$ (b) $y = \dfrac{(x-a)(x+a)}{x}$

(c) $y = \dfrac{(x-a)(x^2+a)}{x^2}$ (d) $y = \dfrac{(x-a)(x+a)}{(x-b)(x+b)}$

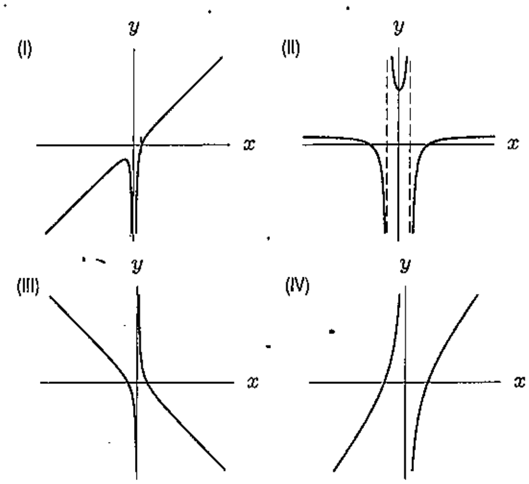

Figure 1.67

30. Find a calculator window in which the graphs of $f(x) = x^3 + 1000x^2 + 1000$ and $g(x) = x^3 - 1000x^2 - 1000$ appear indistinguishable.

31. Use a graphing calculator or a computer to graph $y = x^4$ and $y = 3^x$. Determine approximate domains and ranges that give each of the graphs in Figure 1.68.

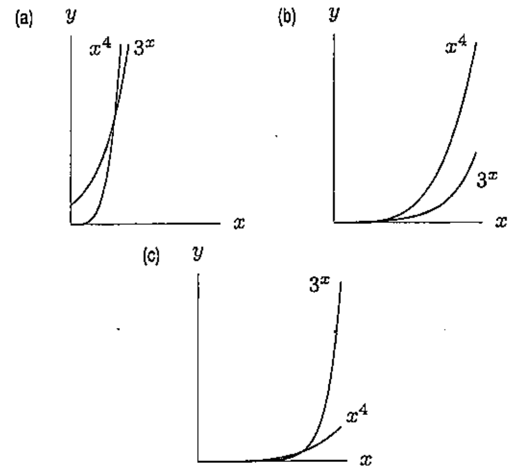

Figure 1.68

32. Newton's Second Law of Motion, $F = ma$, tells us that the net force, F, on a train of mass m is proportional to its acceleration, a. Suppose that the only forces are those of the engine, which exerts a constant force, F_E, in the direction of motion, and the wind resistance, which exerts a force proportional to the square of the train's velocity, v, but in the opposite direction.

(a) Write a formula giving a as a function of v.
(b) Sketch a graph of a against v.

1.7 INTRODUCTION TO CONTINUITY

This section introduces the idea of *continuity* on an interval and at a point. This leads to the concept of limit, which is investigated in the next section.

Continuity of a Function on an Interval: Graphical Viewpoint

Roughly speaking, a function is said to be *continuous* on an interval if its graph has no breaks, jumps, or holes in that interval. Continuity is important because, as we shall see, continuous functions have many desirable properties.

For example, to locate the zeros of a function, we often look for intervals where the function changes sign. In the case of the function $f(x) = 3x^3 - x^2 + 2x - 1$, for instance, we expect[8] to find a zero between 0 and 1 because $f(0) = -1$ and $f(1) = 3$. (See Figure 1.69.) To be sure that

[8]This is due to the Intermediate Value Theorem, which is discussed on page 46.