Lesson: _____

Section: 8.1

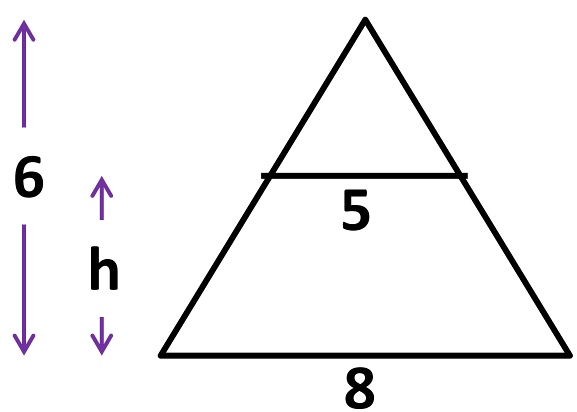
Using the Definite Integral to Calculate Areas & Volumes (Slice Method)

In chapter 5, we calculated areas using definite integrals. We obtained the integral by slicing a region up into rectangles and then summing up all the slices using a Riemann Sum. By taking the limit as $n \to \infty$, we were able to find the exact value of the area.

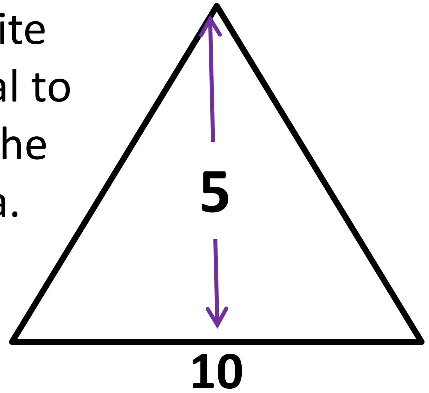
We are now going to calculate VOLUME in a similar fashion using "The Slice Method."

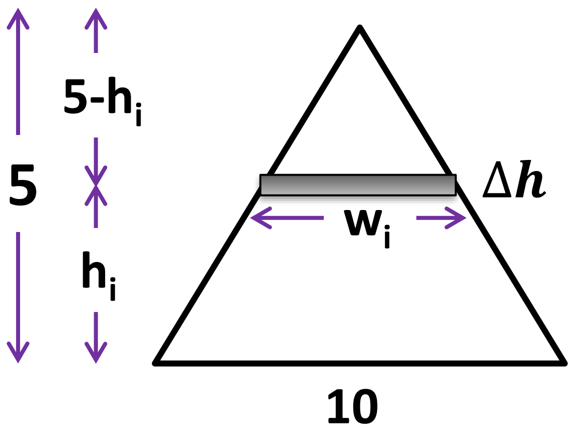
Old School Geometry Warmup

Find h



Use a definite integral to find the area.





Imagine slicing the triangle horizontally, and then summing up all the slices.

←2D Homer

Area of one slice $\approx w_i \, \Delta h$

Now, we'd like to to express everything in terms of one variable. ***Let's get w in terms of h.***

$$\frac{w_i}{10} = \frac{5 - h_i}{5}$$

← Find a relationship between the variables (usually geometry)

$$w_i = 2(5 - h_i)$$
$$w_i = 10 - 2h_i$$

We have now modeled **the area of the slice using only one variable**

Area of one slice $\approx (10 - 2h_i)\Delta h$

Area of the triangle $\approx \sum_{i=1}^{n} (10 - 2h_i)\Delta h$

To find the area of the ***triangle***, let's just sum up all these little slices!

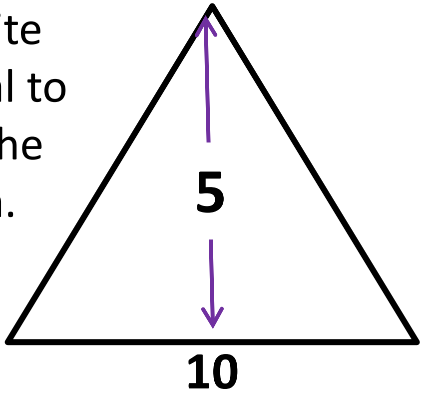
Now, take the limit as $n \to \infty$ $(or\ as\ \Delta h \to 0)$ and we get…

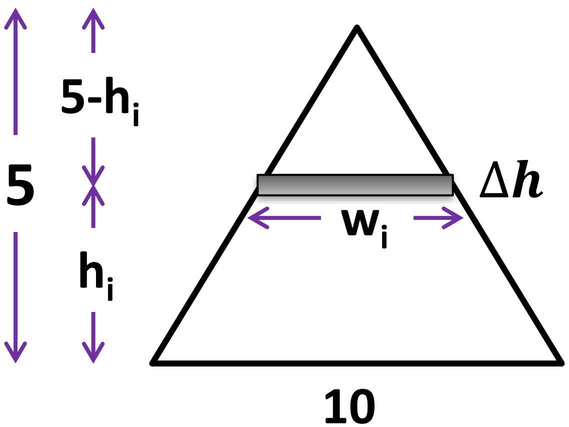
Area of the triangle $= \int_0^5 (10 - 2h)\, dh$

$$\left[10h - h^2\right]_0^5 = [50 - 25] - [0 - 0]$$

$$= 25\ cm^2$$

Use a definite integral to find the area.





Imagine slicing the triangle horizontally, and then summing up all the slices.

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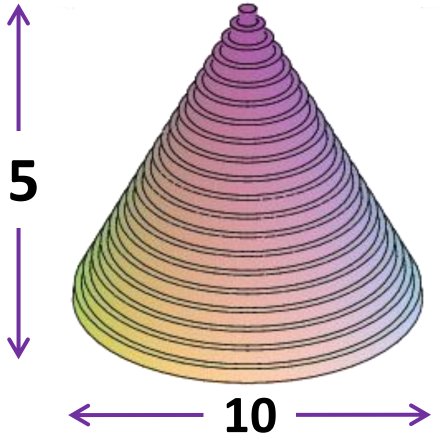
Area of the triangle $= \int_0^5 (\mathbf{10 - 2}\boldsymbol{h})\, dh$

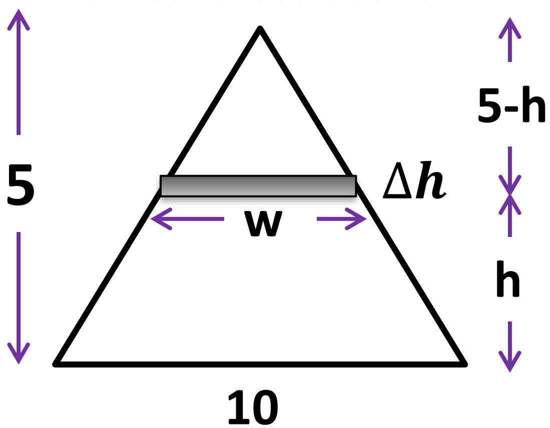
$$\left(10h - h^2\right)_0^5 = [50 - 25] - [0 - 0]$$

$$= 25\ cm^2$$

Use a definite integral to find volume of the cone.



vertical cross-section



Each slice is a circular disk like a coin.

←3D Homer

Volume of one slice $\approx \pi r^2 \, \Delta h$

Now, we'd like to express everything in terms of one variable. ***Let's get r in terms of h.***

$$r = \frac{1}{2}w = \frac{1}{2}(10 - 2h) = 5 - h$$

From example 1

Volume of one slice $\approx \pi(5-h)^2 \, \Delta h$

Note: only 1 variable!

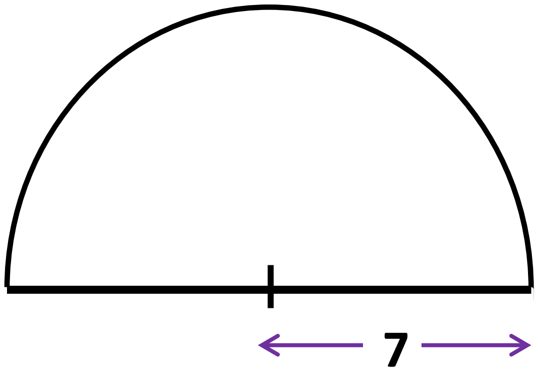
To find the volume of the cone, let's just sum up all these little slices (using a definite integral)!

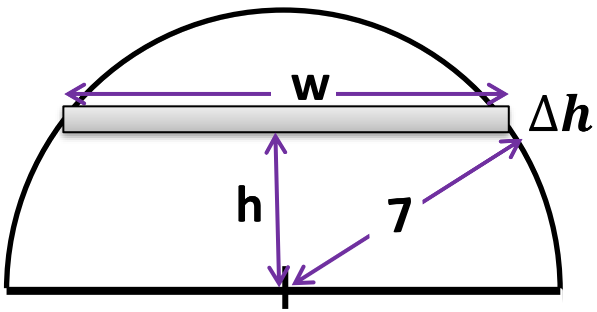
Volume of the cone $= \int_0^5 \pi(5-h)^2 \, \boldsymbol{dh}$

$$\left[-\frac{\pi}{3}(5-h)^3\right]_0^5 = (0) - \left(\frac{-125\pi}{3}\right)$$

$$= \frac{125\pi}{3} \; cm^3$$

Use a definite integral to find the area.





$$h^2 + \left(\frac{w}{2}\right)^2 = 7^2$$

$$\left(\frac{w}{2}\right)^2 = 49 - h^2$$

$$w = 2\sqrt{49 - h^2}$$

Area of one slice $= w\,\Delta h$

Now, express everything in terms of one variable. ***Let's get w in terms of h.***

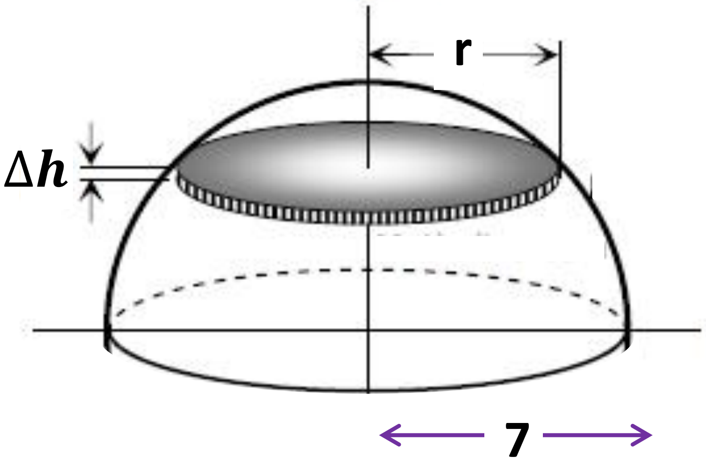
Area of one slice $= \left(2\sqrt{49 - h^2}\right)\Delta h$

Area of the semi-circle $= \displaystyle\int_0^7 2\sqrt{49 - h^2}\,dh$

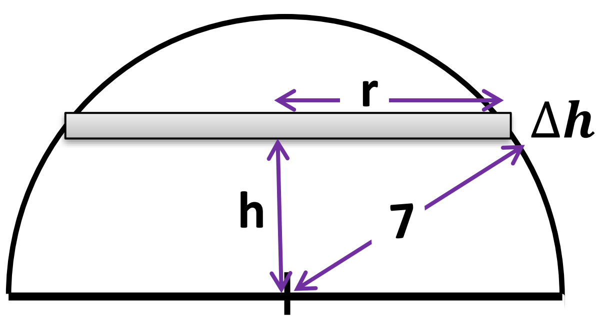
Use the calculator!

Area of the semi-circle $\approx 76.969\ cm^2$

Use a definite integral to find volume of the hemisphere.



vertical cross-section



$$h^2 + r^2 = 7^2$$

$$r^2 = 49 - h^2$$

$$r = \sqrt{49 - h^2}$$

Volume of one slice $= \pi r^2 \, \Delta h$

Volume of one slice $= \pi \left(\sqrt{49 - h^2} \right)^2 \Delta h$

$= \pi(49 - h^2)\Delta h$

Volume of the Solid $= \displaystyle\int_0^7 \pi(49 - h^2)\,dh$

$$\pi \left[49h - \frac{h^3}{3} \right]_0^7 = (343\pi) - \left(\frac{343\pi}{3} \right)$$

Volume of the solid $\approx \mathbf{718.378}\ cm^3$

Warmup

Find the area of the washer if the hole has a diameter of 2 and the washer has a diameter of 6.

Lesson: ______
Section 8.2
Volumes of Solids of Revolution

To Compute a Volume Using an Integral

- Divide the solid into small pieces whose volume we can easily approximate
- Add the contributions of all the pieces, obtaining a Riemann sum that approximates the total volume
- Take the limit as the number of terms in the sum tends to infinity, giving a definite integral for the total volume.

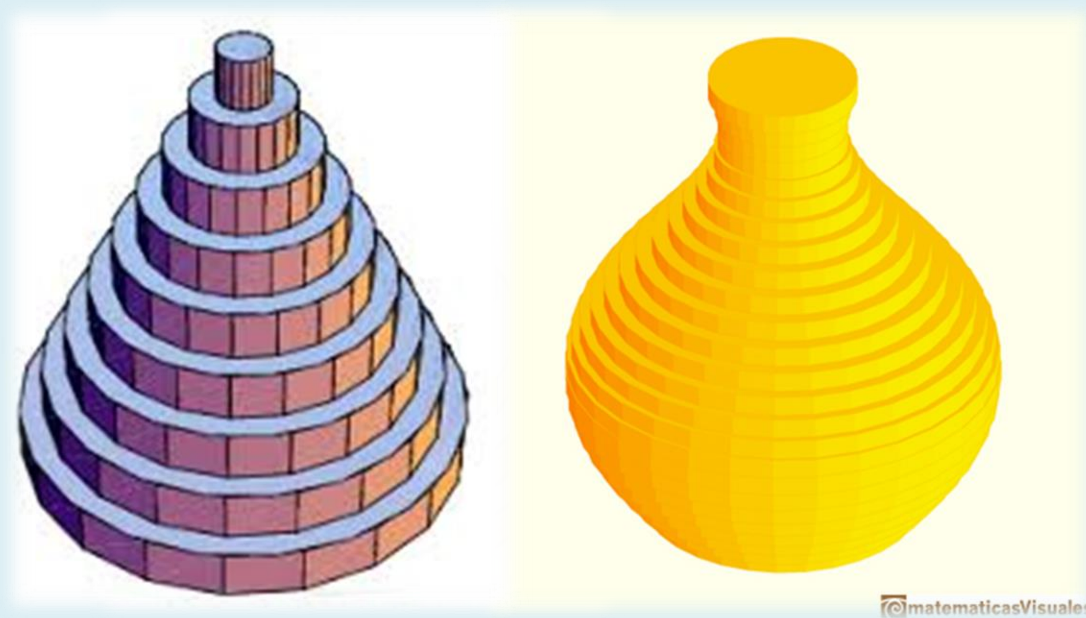
So we model the volume of one slice, then use an integral to accumulate the slices!

Solids of Revolution

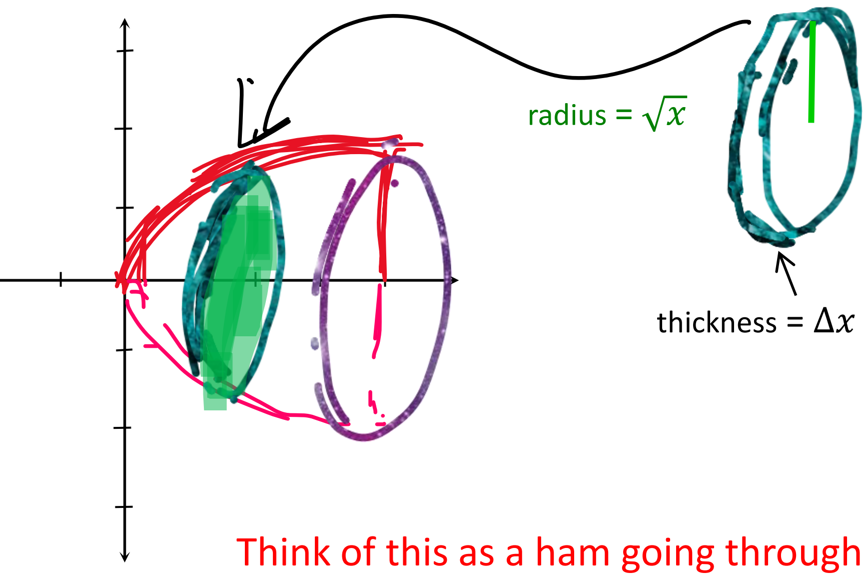
http://demonstrations.wolfram.com/SolidsOfRevolution/
Wolfram Solids of Revolution, 2 examples that I can spin the 3d graph and look

THE "DISK METHOD"

See ex.1,2 in the text



Ex. Take the region bounded by the curve $y = \sqrt{x}, y = 0$, and $x = 4$ and revolve it around the x-axis. Find the volume of the resulting solid.



Think of this as a ham going through a meat slicer at the deli. Each “slice” is a very thin cylinder! If we could model the volume of one of these slices, we could integrate them all to find the whole ham!

Volume of a cylinder

$$V_{slice} = \pi r^2 h = \pi\left(\sqrt{x}\right)^2 \Delta x$$

$$V_{slice} = \pi x \Delta x$$

When we integrate, we are making the slices infinitesimally thin!

$$V_{solid} = \int_0^4 (\pi x)dx$$

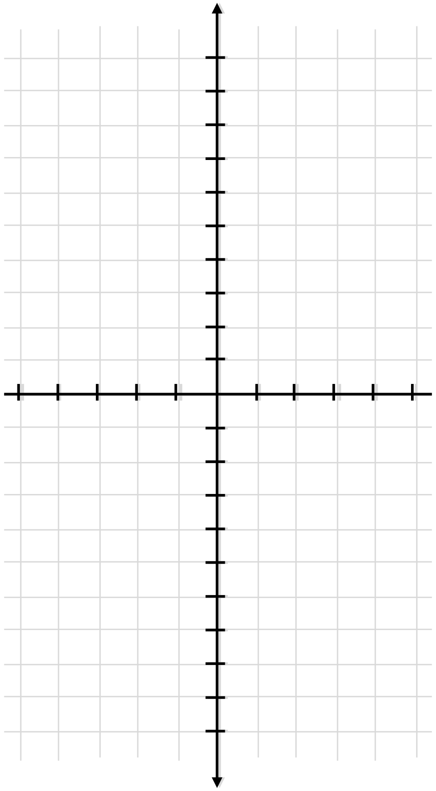
$$= \pi \int_0^4 (x)dx$$

$$= \pi \left[\frac{1}{2}x^2\right]_0^4$$

$$= \pi[8 - 0] = 8\pi \ \ units^3$$

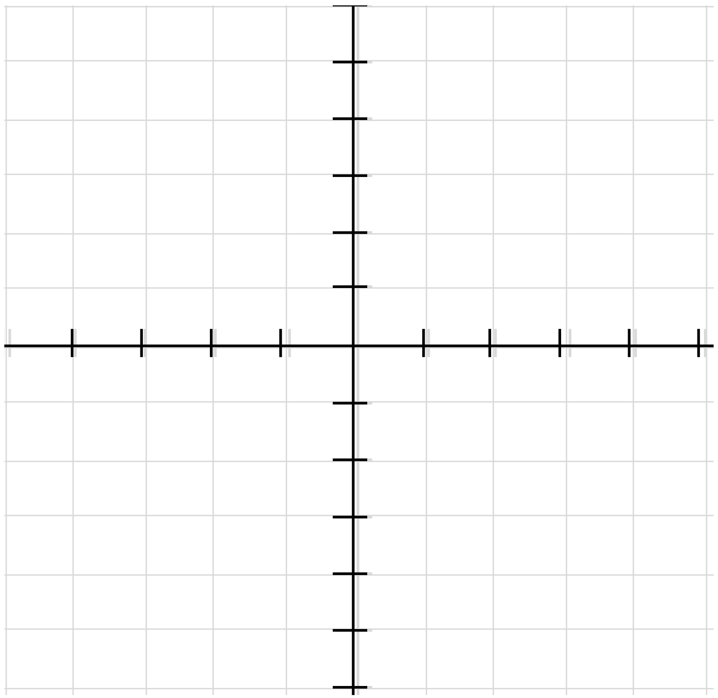
“Disk Method”

Ex. Take the region bounded by the curve $y = x^3$, $\mathrm{y} = 0$, and $\mathrm{x} = 0$, and $\mathrm{y} = 8$ and revolve it around the y-axis. Find the volume of the resulting solid.



“Disk Method”

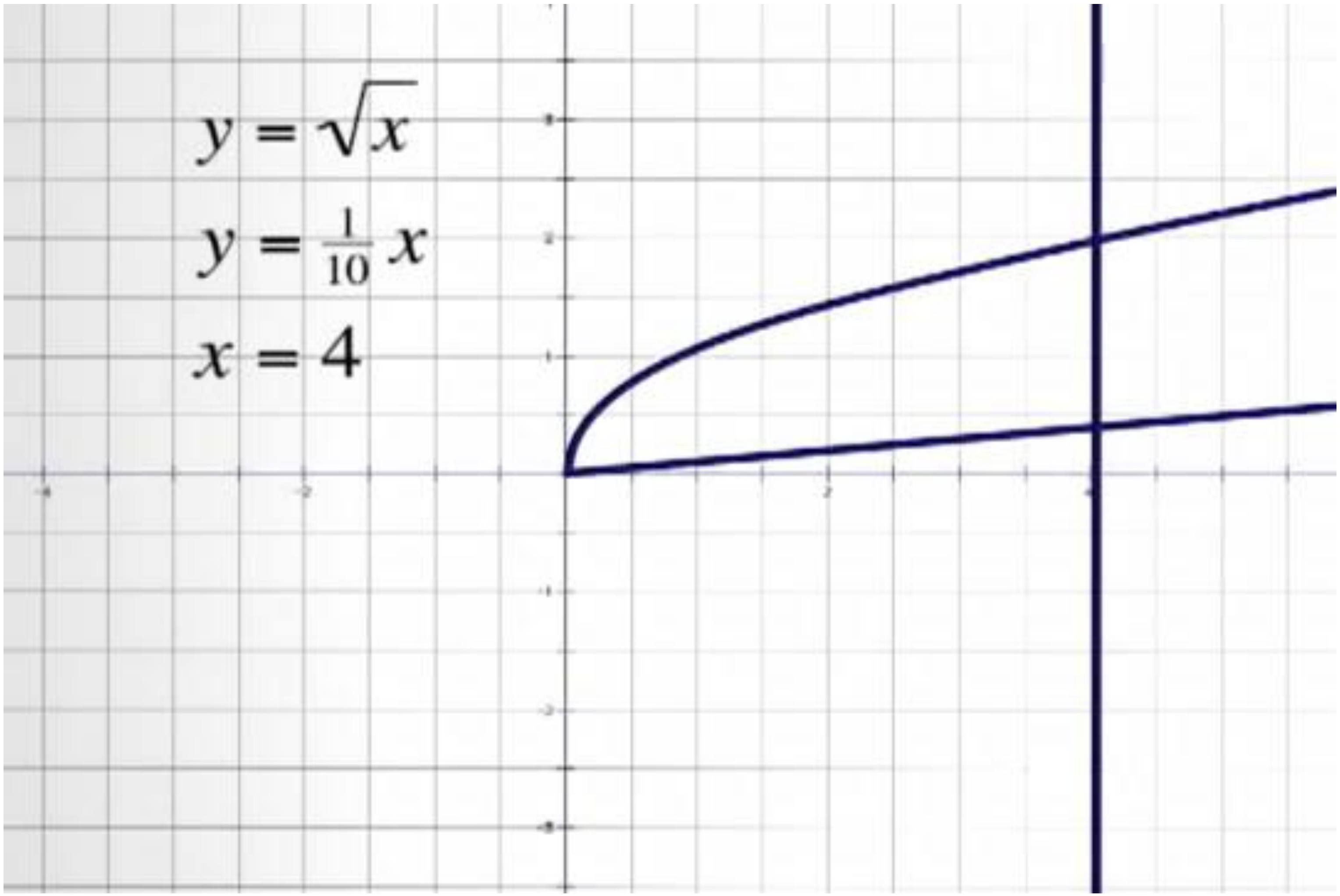
Ex. Take the region bounded by the curve $y = x^2$ and $y = 2x$. and revolve it around the x-axis. Find the volume of the resulting solid.



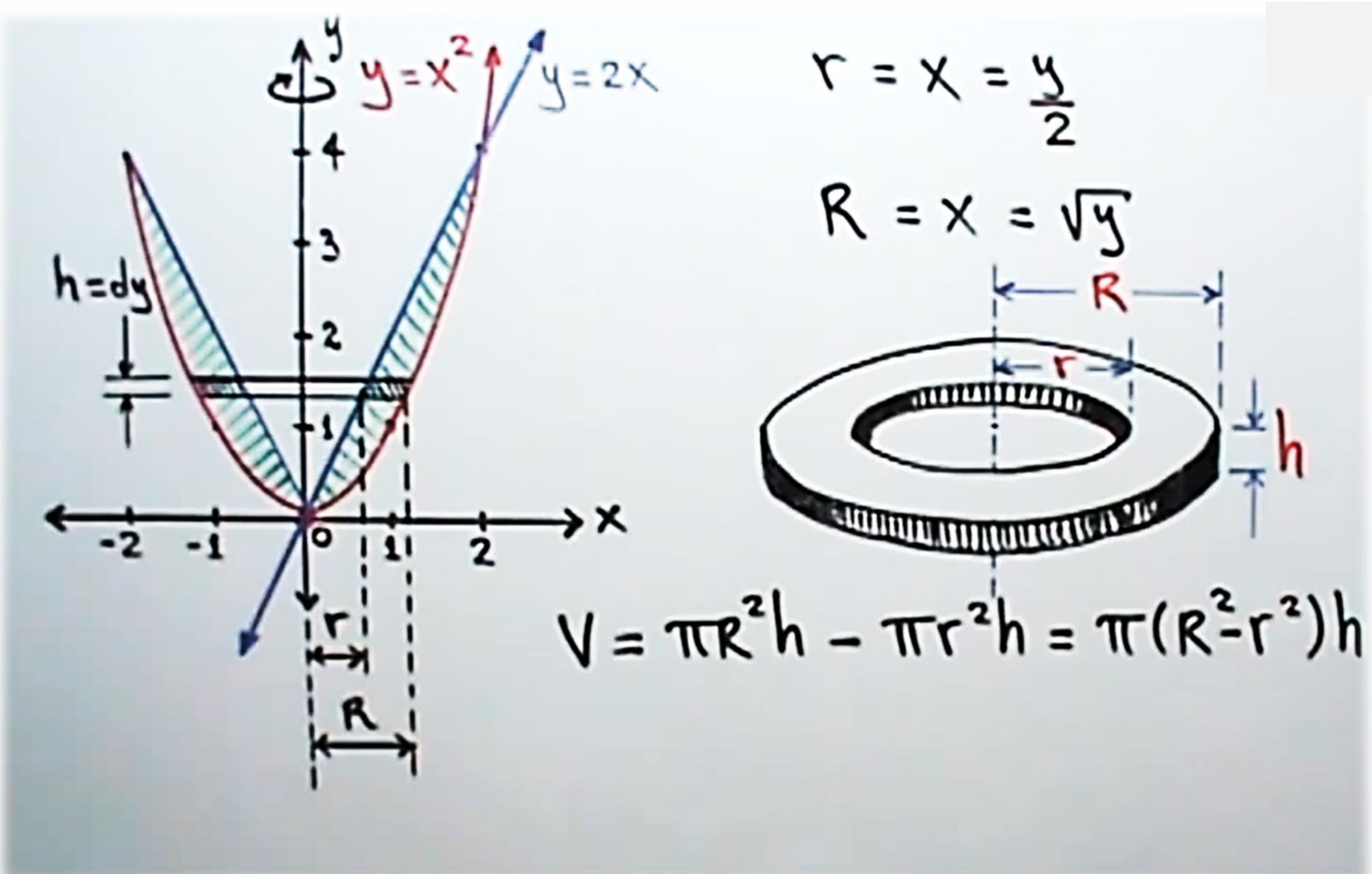
Discovering the “Washer Method”

Visualizing the "Washer Method"

Video
30 sec.

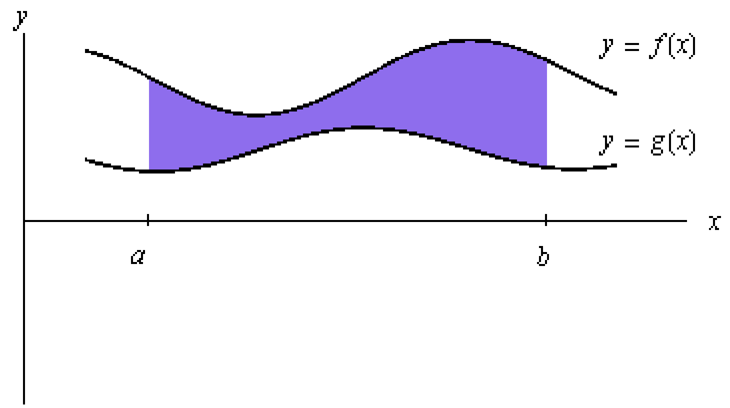


Volumes of Solids of Revolution: The "Washer Method"



http://demonstrations.wolfram.com/SolidOfRevolution/

Nice interactive solid of revolution for a difference of two generic functions (below)



1. **Write an integral to represent the volume of the solid formed by revolving the region around the x-axis.**

$$\int_a^b \pi \left(f(x)^2 - g(x)^2\right) dx$$

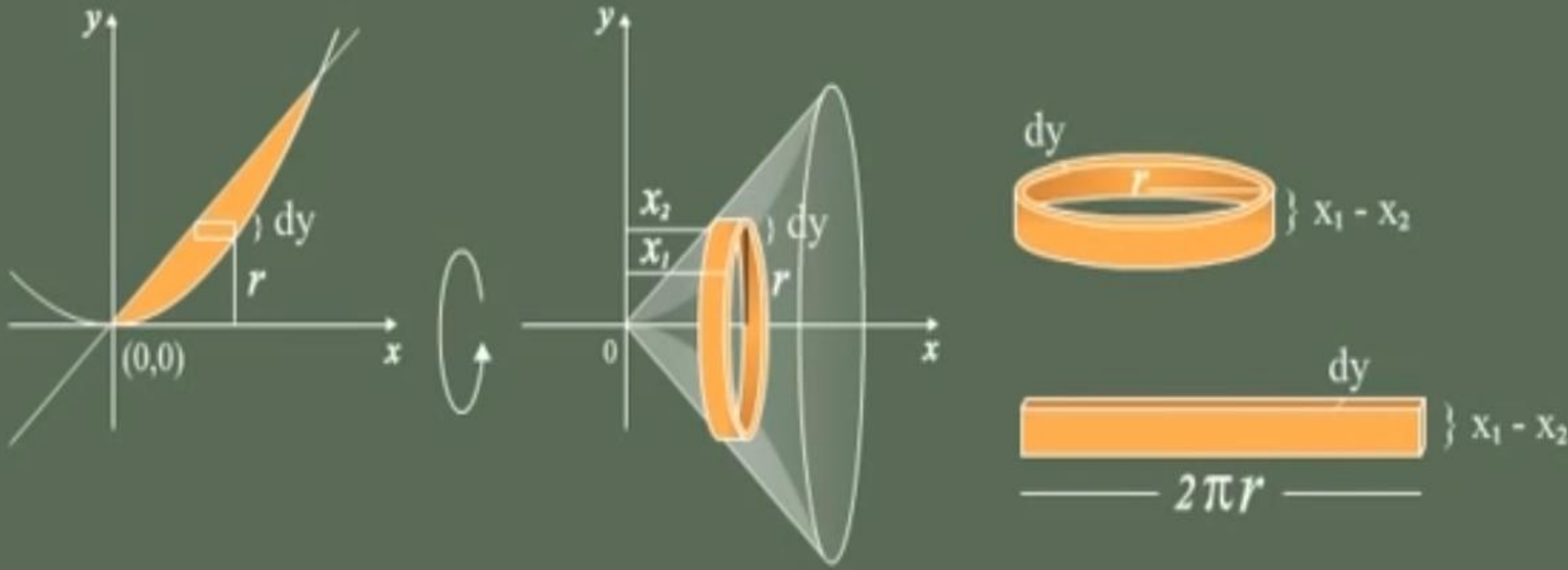
2. **What if we decided to revolve it around the y-axis? Would washers still work? If not, how could we approach this?**

The Shell Method

Video
3 min 38 sec.

Find the volume of the solid formed by rotating about the y axis the region bounded by $y = x^2$ and $y = \sqrt{x}$

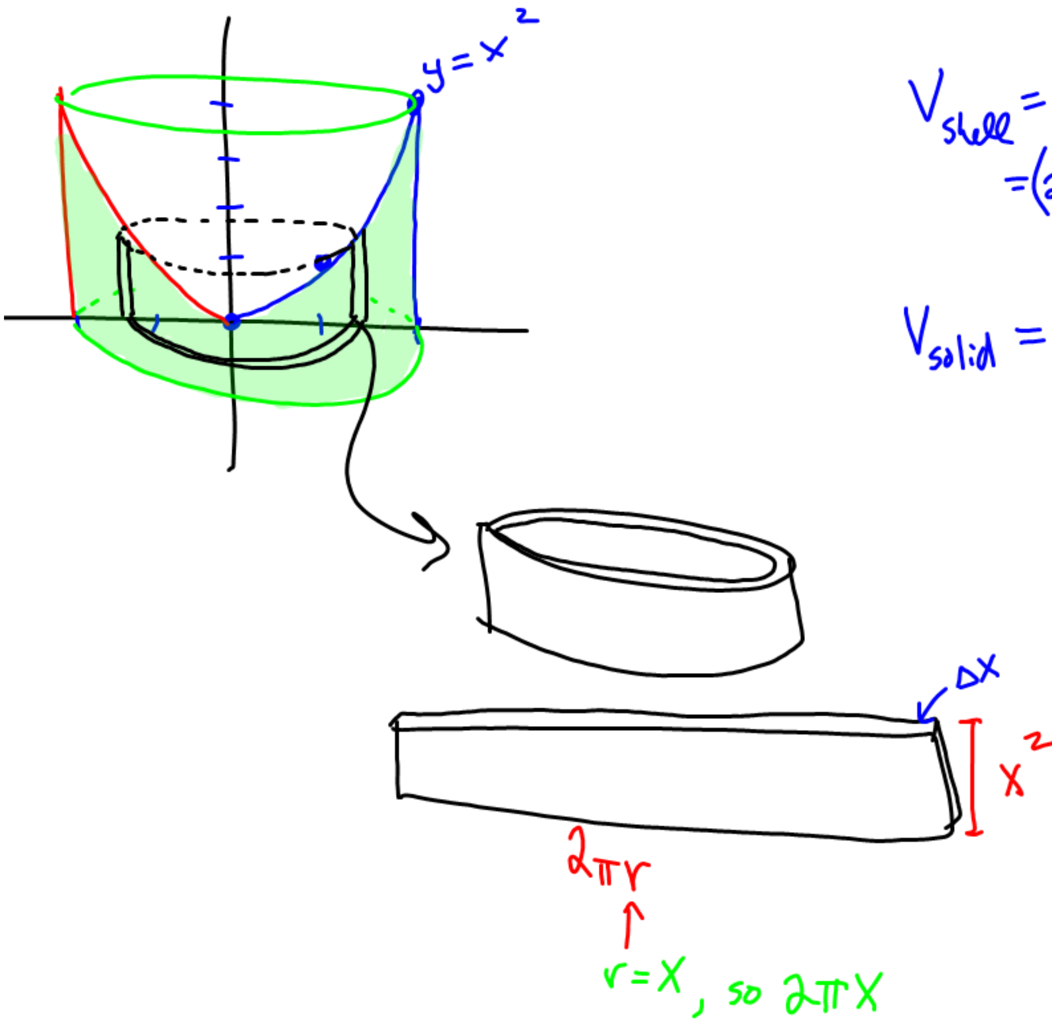
http://www.math.tamu.edu/~tkiffe/calc3/revolution2/revolution2.html
I can animate these to demo what the disc and shell method are doing.



$$dV = 2\pi r\,(x_1 - x_2)\,dy \qquad V = \int_0^1 2\pi r\,(x_1 - x_2)\,dy$$

Shells Worksheet

1.) Use elements parallel to the axis of revolution (that is, use "shells") to find the volume of the solid generated by *revolving about the y-axis* the region bounded by $y = x^2, x = 2,$ and the $x - axis$.

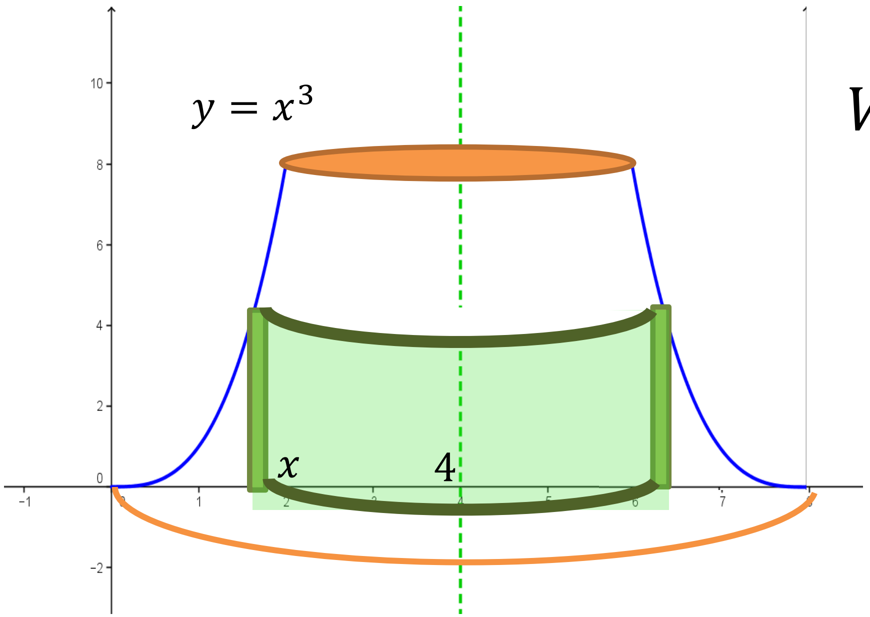


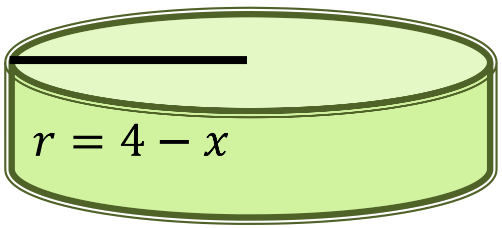
$$V_{shell} = l \cdot w \cdot h = (2\pi x)(x^2)\,\Delta x$$

$$V_{solid} = 2\pi \int_0^2 x^3\,dx = 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{4}\right) = 8\pi$$

Shells Worksheet

2.) Use shells to find the volume of the solid generated by revolving about the line $x = 4$ the region bounded by $y = x^3$ and the lines $y = 0$ and $x = 2$.





$$V_{shell} = lwh = [2\pi(4 - x)]\left(x^3\right)\Delta x$$

$$V = 2\pi \int_2^0 (4 - x)\left(x^3\right) dx$$

$$V = 2\pi \int_2^0 \left(4x^3 - x^4\right) dx$$

$$V = -2\pi \int_0^2 \left(4x^3 - x^4\right) dx$$

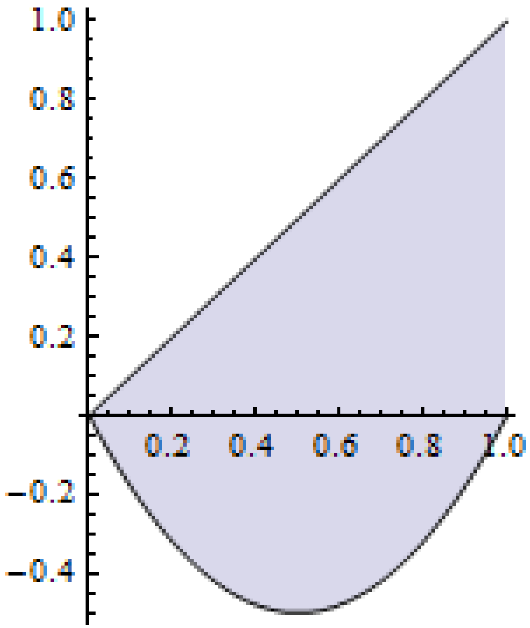
$$V = -2\pi \left[x^4 - \frac{1}{5}x^5\right]_0^2$$

$$V = \qquad units^3$$

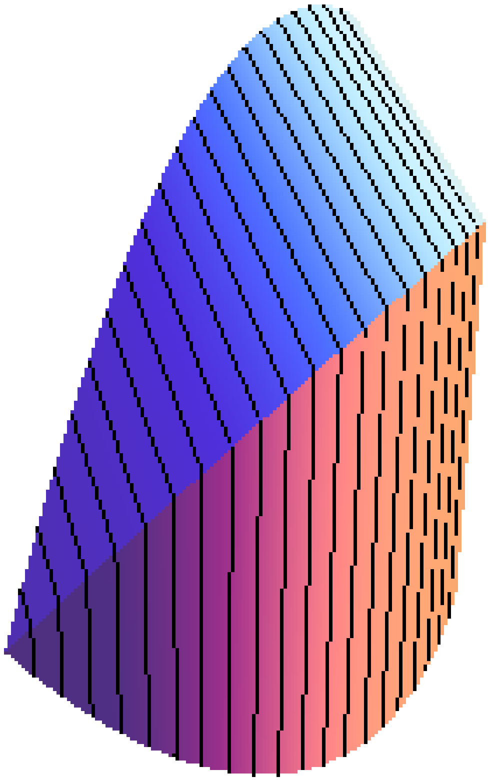
Solids with Known Cross-Sections

http://demonstrations.wolfram.com/SolidsOfKnownCrossSection/
Visualizing solids with known cross sections

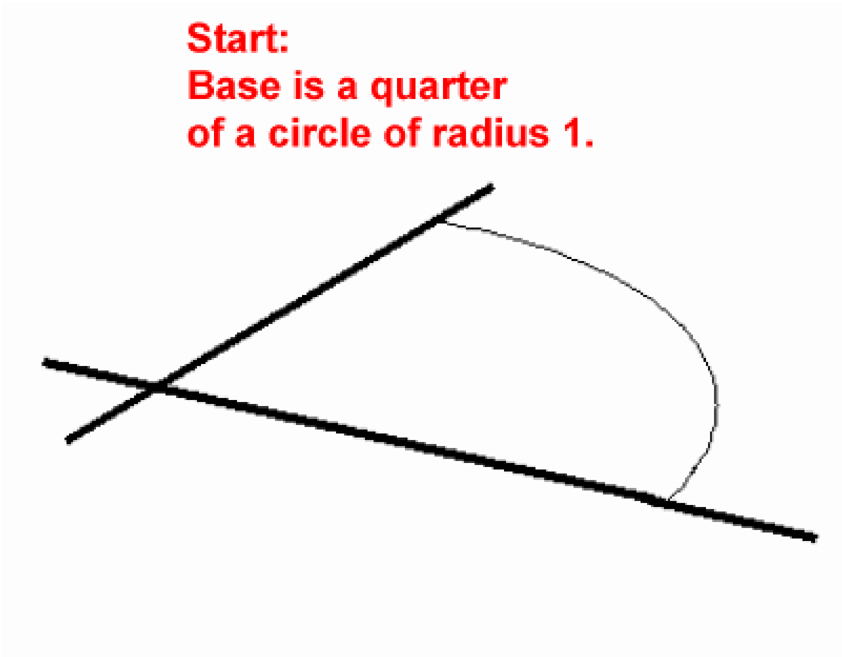
Ex. Find the volume of the solid whose base is the region in the xy-plane bounded by the curves shown and whose cross-sections perpendicular to the x-axis are squares with one side in the xy-plane.

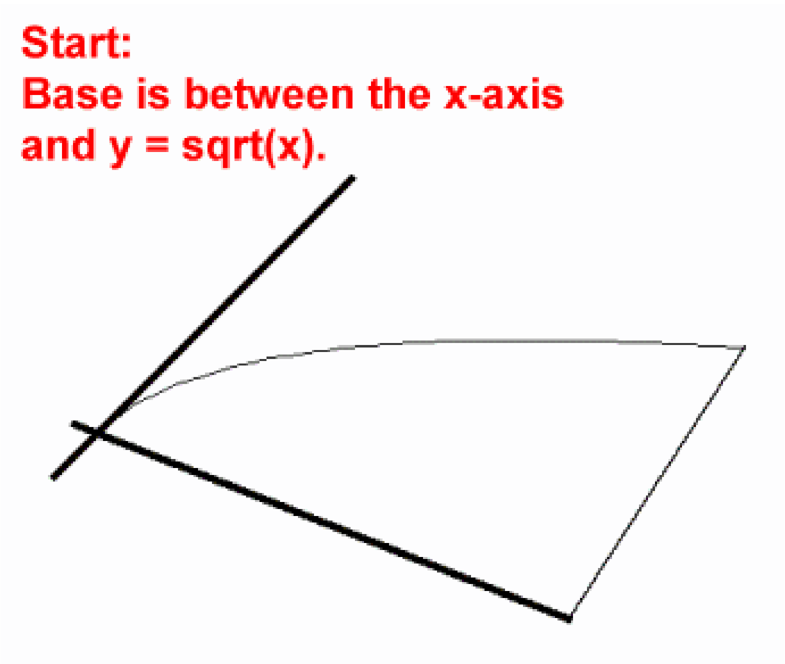


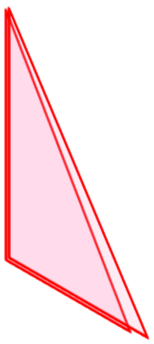
base region

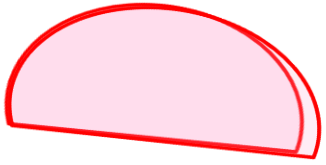


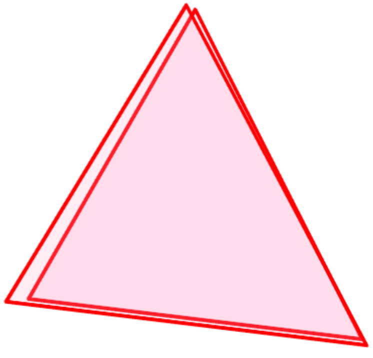
Animation will start in a couple seconds…



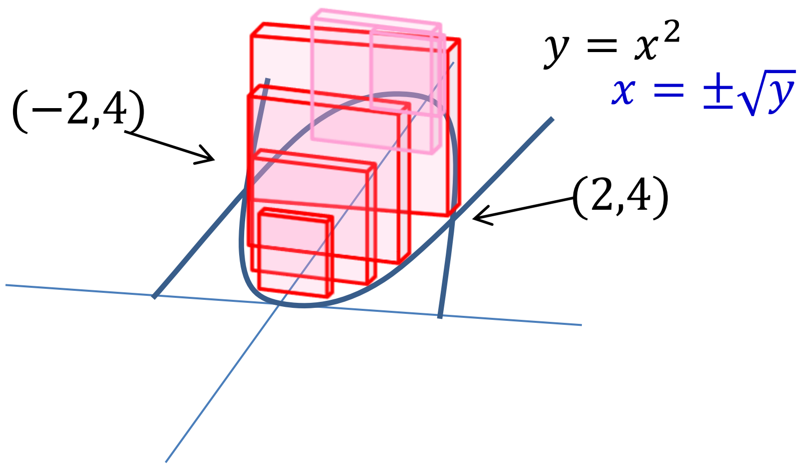


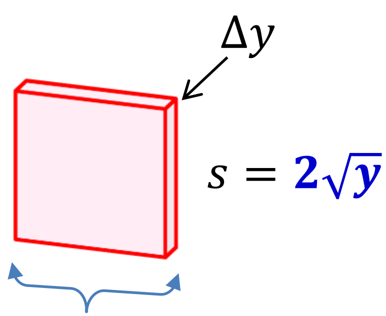






Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y = x^2$ *and* $y = 8 - x^2$ and whose cross-sections perpendicular to the y-axis are squares with one side in the xy-plane.





s= the difference between the x-values

What are these x-values?

$\boldsymbol{x = \sqrt{y} \ \ and \ x = -\sqrt{y}}$

$$V_{slice} = s^2 h = (2\sqrt{y})^2\,\Delta y$$

$$V_{slice} = 4y\,\Delta y$$

$$V_{\substack{bottom\\half}} = 4\int_0^4 y\,dy$$

$$V_{total} = 2 * \ 4\int_0^4 y\,dy$$

$$V = 8\left[\frac{1}{2}y^2\right]_0^4$$

$$V = 8 * 8 = 64\,units^3$$

The washer method explained en Español just for fun. Interdisciplinary instruction!